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MODELING DEFAULTABLE BONDS WITH MEAN-REVERTING LOG-NORMAL SPREAD: A QUASI CLOSED-FORM SOLUTION

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Abstract. In this paper we describe a two factor model for a defaultable discount bond, assuming a mean reverting log-normal dynamics with bounded volatility for the instantaneous short rate spread. Under some simplifying assumptions we obtain an explicit solution for zero recovery in terms of the confluent hypergeometric functions.

1 INTRODUCTION

The approaches to model credit risk can be broadly classified in two classes. The earlier includes the so called structural models, based on the firm's value approach introduced in Merton (1974), and extended in Black and Cox (1976), Longstaff and Schwartz (1995), and others.

More recent is the class of the generally termed as reduced-form models, in which the assumptions on a firm's value are dropped, and the default is modeled as an exogenous stochastic process. Reduced-form models have been proposed in Jarrow and Turnbull (1996), Duffie and Kan (1996), Jarrow et al. (1997), Schonbucher (1998), Cathcart and El Jahel (1998), Duffie and Singleton (1999), Duffie et al. (2000), Schonbucher (2000) and others.

In this paper we present a two factor model that extends the results in Cortina (2001) and Cane de Estrada et al. (2005), where the price of a risky bond price was derived as a function of the risk-free short rate and the instantaneous short spread, with the requirement that the short spread must be positive. This extension is motivated by a remark in Duffie and Kan (1996), saying that the observed empirical behaviour of instantaneous risk of default is mean reverting under the real measure. Therefore, we assume here that the spread follows a mean reverting log-normal random walk with a lower barrier, and that the default occurs if it hits an upper barrier; this last hypothesis is equivalent to assume a bounded volatility process for the dynamics of the spread.

The model presented in Cathcart and El Jahel (1998) is also a reduced-form one, solved by a structural approach that leads to a barrier-type solution; they assume that the default occurs when a signaling process hits some predefined lower barrier, but they do not identify the signaling variable. In the same line as Cathcart and El Jahel, Ho and Hui (2000) propose the foreign exchange rate as a signaling variable and a barrier that is an exponential function of the volatility. This particular characterization of the signal process is not useful for emerging sovereign issuers where the currency is pegged to the dollar by law (e.g., Argentina during the nineties).

The remainder of the paper is organized as follows. The bond pricing equation is derived in Section 1. Section 2. contains the model of the spread. In Section 3. it is shown that the problem can be turned into a Sturm Liouville one and a quasi close solution can be obtained in terms of the hypergeometric functions.

2 THE PRICING EQUATION

We work in a continuous time framework, in which $r_d(t)$ is the defaultable short rate if a default event has not occurred until t, $r(t)$ is the risk-free short rate, and the spread $h(t)$ is defined as

$$h(t) = r_d(t) - r(t).$$

Our assumptions are

1. the dynamic of $r(t)$ and $h(t)$ are governed by diffusion equations

$$dr(t) = \mu_r(r,t)dt + \sigma_r(r,t)dW_1, \tag{1}$$

$$dh(t) = \mu_h(h,t)dt + \sigma_h(h,t)dW_2, \tag{2}$$

where W_1 and W_2 are uncorrelated standard Brownian motions,

2. the spread $h(t) > 0$ is positive.

Using an extension of the Black and Scholes option pricing technique we derive the general pricing equation for a defaultable discount bond,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_r^2(r,t)\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\sigma_h^2(h,t)\frac{\partial^2 P}{\partial h^2} + \phi(r,t)\frac{\partial P}{\partial r} + \psi(h,t)\frac{\partial P}{\partial h} - rP = 0. \tag{3}$$

As long as r and h were not correlated, the problem is separable; i.e. we consider a solution

$$P(r,h,t,T) = Z(r,t,T)S(h,t), \tag{4}$$

where $Z(r,t,T)$ is the solution of a risk free bond. Replacing this solution in (3) gives

$$Z\left[\frac{\partial S}{\partial t} + \frac{1}{2}\sigma_h^2(h,t)\frac{\partial^2 S}{\partial h^2} + \psi(h,t)\frac{\partial S}{\partial h}\right] + S\left[\frac{\partial Z}{\partial t} + \frac{1}{2}\sigma_r^2(r,t)\frac{\partial^2 Z}{\partial r^2} + \phi(r,t)\frac{\partial Z}{\partial r} - rZ\right] = 0,$$

where the second bracket is zero (since it is the solution of the risk-free bond). Then $S(h,t)$ satisfies

$$\frac{\partial S}{\partial t} + \frac{1}{2}\sigma_h^2(h,t)\frac{\partial^2 S}{\partial h^2} + \psi(h,t)\frac{\partial S}{\partial h} = 0, \tag{5}$$

where

$$\psi(h,t) = \mu_h(h,t) - \sigma_h(h,t)\lambda_h(h,t),$$

and $\lambda_h(h,t)$ is the market price of the risk associated with the spread.
If a default has not occurred before the maturity T, the final condition is

$$P(r,h,T,T) = Z(r,T,T)S(h,T) = 1,$$

which leads to the following final conditions for Z and S

$$Z(r,T,T) = 1, \quad S(h,T) = 1.$$

3 MODELING THE SPREAD

We start by expressing the stochastic process followed by the natural logarithm of the spread $x = \ln h$ as the sum of two components. The first one is considered to be totally predictable, and the second one is a diffusion stochastic process. To be precise, the stochastic differential equation for the log-spread is

$$dx = \theta\left(\kappa - x\right)dt + \sigma_0 dW, \quad \ln H_d \leq x \leq \ln H_u, \tag{6}$$

where κ is a constant reversion level, θ is a constant velocity of reversion, and σ_0 is a positive constant. $x(t)$ has a conditional normal distribution with mean

$$E_0(x) = \kappa + c_1 e^{-\theta t}$$

From Ito's lemma we have the following process for the spread

$$dh = h\left[\theta\left(\kappa - x\right) + \frac{1}{2}\sigma_0\right]dt + \sigma_0 dW \equiv \mu_h dt + \sigma_h dW, \quad H_d \leq h(t) \leq H_u \equiv \mu_h dt + \sigma_h dW, \tag{7}$$

where

$$\sigma_h(h,t) = \sigma_0 \min(H_u, h(t)). \tag{8}$$

It is shown in Heath et al. (1992) that this volatility process gives a finite positive spread process. Replacing in (5) the parameters of the SDE (7) we obtain the PDE for the risk-adjusted price

$$\frac{\partial S}{\partial t} + \frac{1}{2}\sigma_0^2 h^2 \frac{\partial^2 S}{\partial h^2} + \left[\mu_h - \lambda_0\sigma_0\right]\frac{\partial S}{\partial h} = 0, \tag{9}$$

where

$$\mu_h - \lambda_0\sigma_0 = h\left(\kappa\theta + \frac{1}{2}\sigma_0^2 - \lambda_0\sigma_0 - \theta x\right), \tag{10}$$

and the integration domain is

$$0 \leq t < T, \quad H_d \leq h(t) \leq H_u.$$

By changing to the dimensionless variables

$$\tau = (T-t)\frac{1}{\theta}, \;\; y = \sqrt{\frac{2\theta}{\sigma_0^2}}\left(\kappa - \frac{\sigma_0\lambda_0}{\theta} - x\right) \tag{11}$$

we can rewrite (9) as

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2} - y\frac{\partial v}{\partial y} = 0, \quad 0 \leq t < T, \quad H_d \leq h(t) \leq H_u. \tag{12}$$

If a default has not occurred until maturity, the condition on the price at maturity $\tau = 0$ is $v(y,0) = 1$. A default occurs whenever $h = H_u$, i.e. $v(y_u,\tau) = 0$. Assuming that the prices tend to stability when h drops to a fixed value H_d, we have a third condition, namely $\frac{\partial v}{\partial y}(y_d,\tau) = 0$. Summarizing, the three conditions are

$$v(y,0) = 1, \tag{13}$$

$$v(y_u,\tau) = 0, \tag{14}$$

$$\frac{\partial v}{\partial y}(y_d,\tau) = 0. \tag{15}$$

A solution

$$v = e^{\lambda\tau}u(y) \tag{16}$$

separates the problem (12), and leads to

$$u'' - yu - \lambda u = 0. \tag{17}$$

The linearly independent solutions to (12) are

$$u_1 = y\Phi\left(\frac{\lambda+1}{2}, \frac{3}{2}, \frac{y^2}{2}\right), \tag{18}$$

$$u_2 = \Phi\left(\frac{\lambda}{2}, \frac{1}{2}, \frac{y^2}{2}\right), \tag{19}$$

given in terms of the confluent hypergeometric function

$$\Phi(a,b,z) = 1 + \frac{az}{b} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots \tag{20}$$

For integer $a < 0$, Φ is a polynomial. The eigenvalues λ_n are the roots of the determinant of the system

$$Au_1(y_u) + Bu_2(y_u) = 0,$$

$$Au_1'(y_d) + Bu_2'(y_d) = 0,$$

and the hypergeometric function is derived from

$$\frac{\partial}{\partial y}\Phi\left(a, b, \frac{y^2}{2}\right) = \frac{ay}{b}\Phi\left((a+1, b+1, \frac{y^2}{2}\right).$$

The boundary conditions (14) and (15) should be explicitly imposed on a linear combination of u_1 and u_2, for any value of λ and, since they are homogeneous, they determine the weight of u_1 relative to u_2 for the same eigenvalue λ.

4 THE STURM LIOUVILLE PROBLEM

The transformation

$$u = e^{y^2/4}\Psi(y)$$

changes the equation (17) into the normal form

$$\Psi'' + \frac{1}{2}\left(1 - 2\lambda - \frac{y^2}{2}\right)\Psi = 0. \tag{21}$$

This equation, together with the boundary conditions for Ψ, constitutes a Sturm-Liouville problem. Two solutions Ψ_1, Ψ_2 corresponding to different eigenvalues $\lambda_{1,2}$ are orthogonal since

$$[\Psi_2\Psi_1' - \Psi_1\Psi_2'] = (\lambda_1 - \lambda_2)\int_{y_d}^{y_u} \Psi_1\Psi_2 dy = 0, \tag{22}$$

and both solutions in the left side satisfy by construction the same boundary conditions. The orthogonality condition can be written in terms of u functions as

$$\int_{y_d}^{y_u} e^{y^2/2}u_1u_2 dy = 0, \quad \text{for } \lambda_1 \neq \lambda_2. \tag{23}$$

Furthermore, using the normalization conditions the eigenfunctions u_n are normalized such that

$$\int_{y_d}^{y_u} e^{y^2/2} u_1^2 = 1. \tag{24}$$

The solution to the problem

$$v(y,\tau) = \sum_n c_n e^{\lambda_n \tau} u_n(y) \tag{25}$$

is obtained by calculating the coefficients c_n of the expansion in the orthonormal basis from (13)

$$c_n = \int_{y_d}^{y_u} e^{y^2/2} u_n(y) dy, \tag{26}$$

and a solution is completely determined if one only knows the values of y_u and y_d, that can be calculated from $H_{u,d}$ by using (11).

5 CONCLUSIONS

For a two factor model of a defaultable discount bond and modeling the spread as a mean-reverting log-normal random walk with bounded volatility, we have arrived to a Sturm Liouville Problem for the spread from which a quasi closed form solution can be obtained.

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