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THE TRANSITION CONDITIONS IN THE DYNAMICS OF ELASTICALLY RESTRAINED BEAMS AND PLATES.

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Abstract. This paper deals with two problems:
1) The free transverse vibration of a non homogeneous tapered beam subjected to general axial forces, with arbitrarily located internal hinge and elastics supports, and ends elastically restrained against rotation and translation.
2) The free transverse vibration of anisotropic plates of different geometrical, generally restrained boundaries which is restrained against translation along an intermediate line and has an internal hinge elastically restrained against rotation.

A rigorous and complete development is presented. First, a brief description of several papers previously published is included. Second, the Hamilton´s principle is rigorously stated by defining the domain D of the action integral and the space D_a of admissible directions. The differential equations, boundary conditions, and particularly the transitions conditions, are obtained. Third, the transition conditions are analysed for several sets of restraints conditions. Fourth, the existence and uniqueness of the weak solutions of the boundary value problem and the eigenvalue problem which respectively govern the statical and dynamical behaviour of the mentioned mechanical systems is treated. Finally, the method of separation of variables is used for the determination of the exact frequencies and mode shapes and/or a modern application of the Ritz method to obtain approximate eigenvalues. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature have been considered. New results are presented for different boundary conditions and restraint conditions in the internal hinge.

1 INTRODUCTION

Lagrange invented the “operator” δ and with its application a $\delta-$calculus which was viewed as a kind of “higher” infinitesimal calculus. This discipline has attracted the attention of numerous eminent mathematicians, who made important contributions to its development. In the last decades the interest in application of the techniques of the calculus of variations has increased noticeably. This is partly due to the demands of the technology and the availability of powerful computers.

Variational principles have always played an important role in theoretical mechanics. In 1717 Johann Bernoulli presented the principle of virtual work and in 1835 emerged Hamilton´s principle. Particularly, this last principle provides a straightforward method for determining equations of motion and boundary conditions of mechanical systems. Substantial literature has been devoted to the theory and applications of the calculus of variations. For instance, the excellent books, Guelfand and Fomin (1963) and Troutman (1996) present clear and rigorous treatments of the theoretical aspects of the mentioned discipline. Several classical textbooks, Dym and Shames (1973); Kantorovich and Krylov (1964) and Weinstock (1974) present formulations, by means of variational techniques, of boundary value and eigenvalue problems in the statics and dynamics of mechanical systems.

On the other hand, the study of vibration problems of beams and plates with several complicating effects has received considerable treatment. It is not possible to give a detailed account because of the great amount of information, nevertheless some references will be cited. Several investigators have studied the influence of rotational and/or translational restraints at the boundaries, Blevins (1979); Mabie and Rogers (1968); Goel (1976); Hibbeler (1975); Rao and Mirza (1989); Nallim and Grossi (1999); Grossi and Laura (1979); Grossi and Bhat (1995); Mukhopadhyay (1987); Warburton and Edney (1984). Also, the study on vibration of beams with intermediate elastic restraints has been performed by several researchers, Rutemberg (1978); Rao (1989); Grossi and Albarracín (2003).

A review of the literature further reveals that there is only a limited amount of information for the vibration of plates with intermediate restraints and beams with internal hinges. Ewing and Mirsafian (1996) analysed the forced vibrations of two beams joined with a non-linear rotational joint. Wang and Wang (2001) studied the fundamental frequency of a beam with an internal hinge and subjected to an axial force. Chang et al. (2006) investigated the dynamic response of a beam with an internal hinge, subjected to a random moving oscillator. The problem has not been treated in plates.

Modern developments in engineering are making increasing use of several mathematical theories that in the past have been considered as tools of pure mathematicians. A typical method of solving boundary and eigenvalue problems for elliptic partial differential equations with variable coefficients is the variational method.

In most cases of interest in engineering, there exists a variational problem, equivalent to the boundary or eigenvalue problem considered. But the differential equation involves unnecessarily derivatives of higher order than the order of the derivatives included in the corresponding functional which describes certain type of energy. So, it is more natural, from a physical point of view, to look for the weak solution of the given problem than to look for its classical solution, Zeidler(1995a,b); Necas (1967); Rektorys(1980). Since the restrictions on smoothness for weak solutions are milder than those for classical solutions, the variational approach extends the set of problems which can be investigated. Moreover, the classical solution, does not exist for many important engineering and mathematical physics problems.

One of the reasons of the present paper is to present a rigorous procedure, by formulating the stationary condition for the functional $\int_{t_a}^{t_b} \left(T_b - U\right) dt,$ involved in Hamilton´s principle, in the space of admissible functions D and the space D_a of admissible directions. Another motivation is to determine sufficient conditions for the existence and uniqueness of the weak solution of the boundary value problem and of the eigenvalue problem which respectively govern the statical and dynamical behaviour of the mechanical systems under study.

It is also the purpose of the present paper to determine the natural frequencies and the effects of the elastic restraints of the described systems. The method of separation of variables is used for the determination of the exact frequencies and mode shapes. In addition, several cases are solved by the Ritz method with systems of simple polynomials as bases. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature, have been considered, and comparisons of numerical results are included. The algorithms developed can be applied to a wide range of elastic restraint conditions, different material characteristics, step changes in cross-section and in axial force and distributed axial forces. The effects of the variations of the elastic restraints at the boundaries, at the intermediate points or lines and at the internal hinges on the dynamics characteristics are investigated. The transitions conditions are particularly analysed, since these conditions are essential to study the action of an internal hinge with a rotational restraint and the action of intermediate rotational and translational restraints.

Tables and figures are given for frequencies, and in some selected cases, two-dimensional plots for mode shapes are included. A great number of problems were solved and, since this number of cases is prohibitively large, results are presented for only a few cases.

2. BEAMS WITH COMPLICATING EFFECTS

2.1 Variational derivation of the boundary and eigenvalue problems.

Let us consider the tapered beam of length l, which has elastically restrained ends, is constrained at an intermediate point and has an internal hinge elastically restrained against rotation, as shown in Figure 1.

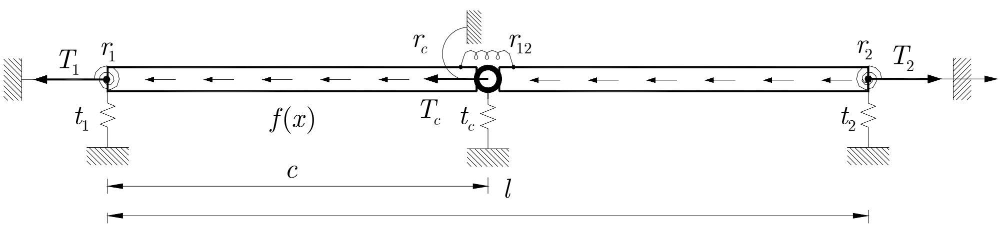


Figure 1: The elastically restrained beam with an internal hinge and intermediate supports.

The beam system is made up of two different spans, which correspond to the intervals $[0,c]$ and $[c,l]$ respectively, with variable mass per unit length and variable flexural rigidity of the ith span as $m_i(x) = \rho_i(x)A_i(x)$ and $D_i(x) = E_i(x)I_i(x)$. It is assumed that the ends, the intermediate point c and the hinge are elastically restrained against translation and/or rotation. The rotational restraints are characterised by the spring constants r_1, r_2, r_{12} and r_c, and the translational restraints by the spring constants t_1, t_2 and t_c. Adopting the adequate values of the parameters r_i and t_i, $i = 1,2,$ all the possible combinations of classical end conditions, (i.e.: clamped, pinned, sliding and free) can be generated. On the other hand, adopting the adequate values of the parameters r_c, r_{12} and t_c different constraints on the point $x = c$ and on the hinge can be generated. It is also assumed that the beam is subjected to axial forces.

In order to analyse the transverse planar displacements of the system under study, we suppose that the vertical position of the beam at any time t is described by the function $u = u(x,t), x \in [0,l]$. It is well known that at time t, the kinetic energy of the beam can be expressed as $T_b = \frac{1}{2}\int_0^c m_1(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\int_c^l m_2(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx.$

Since the beam is subjected to the axial tensile force $T(x)$, the total potential energy due to the elastic deformation of the beam, the springs at the ends restraints and the springs at the intermediate restraints is given by:

$$
\begin{aligned}
U = \frac{1}{2}\Bigg\{ & \int_0^c D_1(x)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 dx + \int_c^l D_2(x)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 dx + \int_0^c T_1(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx \\
& + \int_c^l T_2(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx + r_1\left(\frac{\partial u(0^+,t)}{\partial x}\right)^2 + t_1 u^2(0^+,t) + r_c\left(\frac{\partial u(c^-,t)}{\partial x}\right)^2 \\
& + r_{12}\left(\frac{\partial u(c^+,t)}{\partial x} - \frac{\partial u(c^-,t)}{\partial x}\right)^2 + t_c u^2(c,t) + r_2\left(\frac{\partial u(l^-,t)}{\partial x}\right)^2 + t_2 u^2(l^-,t)\Bigg\},
\end{aligned}
$$

where the notations $0^+, c^-, c^+$ and l^- imply the use of lateral limits and lateral derivatives. It can be observed that the strain energy due to the rotational restraint of coefficient r_c, is computed by means of the expression $\frac{r_c}{2}\left(\frac{\partial u(c^-,t)}{\partial x}\right)^2$, which implies that the spring is connected at right end of the span which corresponds to the interval $[0,c]$, and is connected to a fixed wall. On the other hand, the strain energy which corresponds to the rotational restraint of the internal hinge, is computed by $\frac{r_{12}}{2}\left(\frac{\partial u(c^+,t)}{\partial x} - \frac{\partial u(c^-,t)}{\partial x}\right)^2$, which implies that the spring is connected at right end of the first span and at the left end of the second span.

Calculus of variations is a discipline in which the "operator" δ has been assigned special properties not subsumed in the rigorous formalism of mathematics. A mechanical "δ-method" has been developed and extensively used. In the current engineering literature, it can be observed its use with heuristics developments. This lack of rigor can arise as a disadvantage, but fortunately, it can be easily overcome, since the variation δI of a functional is a straightforward generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$. In the present paper a rigorous formulation of the Hamilton´s principle is presented. The procedure adopted is particularly important in the determination of the analytical expression of the corresponding boundary conditions and transition conditions.

Hamilton´s principle requires that between times t_a and t_b, at which the positions are known, the motion will make stationary the action integral $F(u) = \int_{t_a}^{t_b} L dt$ on the space of admissible functions, where the Lagrangian L is given by $L = T_b - U$, Troutman (1973). In consequence, the energy functional to be considered is given by

$$F(u)=\frac{1}{2}\int_{t_a}^{t_b}\left[\sum_{i=1}^{2}\int_{\Omega_i}\left(m(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2-D(x)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2-T(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right)dx\right]dt$$
$$-\frac{1}{2}\int_{t_a}^{t_b}\sum_{i=1}^{3}a_i\left(\frac{\partial u(b_i,t)}{\partial x}\right)^2dt-\frac{1}{2}\int_{t_a}^{t_b}r_{12}\left(\frac{\partial u(d_2,t)}{\partial x}-\frac{\partial u(b_2,t)}{\partial x}\right)^2dt-\frac{1}{2}\int_{t_a}^{t_b}\sum_{i=1}^{3}c_i u^2(d_i,t)dt, \tag{1}$$

where:

$$m(x)=\begin{cases}m_1(x)=\rho_1(x)A_1(x),\ \forall x\in[0,c),\\ m_2(x)=\rho_2(x)A_2(x),\ \forall x\in(c,l],\end{cases}$$

$$D(x)=\begin{cases}D_1(x)=E_1(x)I_1(x),\ \forall x\in[0,c),\\ D_2(x)=E_2(x)I_2(x),\ \forall x\in(c,l],\end{cases}\qquad T(x)=\begin{cases}T_1(x),\ \forall x\in[0,c),\\ T_2(x),\ \forall x\in(c,l],\end{cases}$$

$\Omega_1=(0,c),\Omega_2=(c,l)$, and the coefficients a_i,b_i,c_i and d_i are defined in the Table 1.

i	a_i	b_i	c_i	d_i
1	r_1	0^+	t_1	0^+
2	r_c	c^-	t_c	c^+
3	r_2	l^-	t_2	l^-

Table 1. Definition of coefficients a_i,b_i,c_i and d_i in Eq. (1).

The stationary condition for the functional (1) requires that

$$\delta F(u,v)=0,\forall v\in D_a. \tag{2}$$

The space D is given by

$$D=\left\{u;u(x,\bullet)\in C^2[t_a,t_b],u(\bullet,t)\in C\left(\overline{\Omega}\right),u(\bullet,t)|_{\overline{\Omega}_i}\in C^4\left(\overline{\Omega}_i\right),i=1,2,\right.$$
$$\left.u(x,t_a),u(x,t_b)\text{ prescribed}\right\}. \tag{3}$$

The only admissible directions v at $u\in D$ are those for which $u+\varepsilon v\in D$ for sufficiently small ε and $\delta F(u;v)$ exists. In consequence, and in view of (3), v is an admissible direction at u for D if, and only if, $v\in D_a$ where

$$D_a=\left\{v;v(x,\bullet)\in C^2[t_a,t_b],v(\bullet,t)\in C\left(\overline{\Omega}\right),v(\bullet,t)|_{\overline{\Omega}_i}\in C^4\left(\overline{\Omega}_i\right),i=1,2,\right.$$
$$\left.v(x,t_a)=v(x,t_b)=0,\forall x\in[0,l]\right\}. \tag{4}$$

Now it is possible to introduce the definition of the variation of F at u in the direction v, as a generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$, Troutman (1973). Consequently, the definition of the first variation of F at u in the direction v, is given by

$$\delta F(u;v)=\left.\frac{dF(u+\varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0} \tag{5}$$

The application of (5) leads to

$$\delta F(u;v)=\int_{t_a}^{t_b}\left[\sum_{i=1}^{2}\int_{\Omega_i}\left(m(x)\frac{\partial u(x,t)}{\partial t}\frac{\partial v(x,t)}{\partial t}-D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\frac{\partial^2 v(x,t)}{\partial x^2}-\right.\right.$$
$$\left.\left.-T(x)\frac{\partial u(x,t)}{\partial x}\frac{\partial v(x,t)}{\partial x}\right)dx\right]dt-\int_{t_a}^{t_b}\sum_{i=1}^{3}a_i\frac{\partial u(b_i,t)}{\partial x}\frac{\partial v(b_i,t)}{\partial x}dt- \tag{6}$$
$$-\int_{t_a}^{t_b}r_{12}\left(\frac{\partial u(d_2,t)}{\partial x}-\frac{\partial u(b_2,t)}{\partial x}\right)\left(\frac{\partial v(d_2,t)}{\partial x}-\frac{\partial v(b_2,t)}{\partial x}\right)dt-\int_{t_a}^{t_b}\sum_{i=1}^{3}c_iu(d_i,t)v(d_i,t)dt.$$

A procedure of integration by parts, transforms (6) in

$$\delta F(u;v)=-\int_{t_a}^{t_b}\left\{\sum_{i=1}^{2}\int_{\Omega_i}\left[m(x)\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right)-\right.\right.$$
$$\left.\left.-\frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right)\right]v(x,t)dx\right\}dt+\int_{t_a}^{t_b}\sum_{i=1}^{2}\left(P_i\frac{\partial v(A_i,t)}{\partial x}+Q_iv(A_i,t)+\right. \tag{7}$$
$$\left.+R_i\frac{\partial v(B_i,t)}{\partial x}+Z_iv(c,t)\right)dt,$$

where

$$P_i=-r_i\frac{\partial u(A_i,t)}{\partial x}+(-1)^{i+1}D(A_i)\frac{\partial^2 u(A_i,t)}{\partial x^2},$$
$$Q_i=-t_iu(A_i,t)+(-1)^i\left[\frac{\partial}{\partial x}\left(D(A_i)\frac{\partial^2 u(A_i,t)}{\partial x^2}\right)-T(A_i)\frac{\partial u(A_i,t)}{\partial x}\right],$$
$$R_i=(-1)^{i-1}\left(r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)-D(B_i)\frac{\partial^2 u(B_i,t)}{\partial x^2}\right)-C_i\frac{\partial u(B_i,t)}{\partial x},$$
$$S_i=-\frac{1}{2}t_cu(c,t)+(-1)^{i-1}\left(\frac{\partial}{\partial x}\left(D(B_i)\frac{\partial^2 u(B_i,t)}{\partial x^2}\right)-T(B_i)\frac{\partial u(B_i,t)}{\partial x}\right),$$
$$A_1=0^+,A_2=l^-,B_1=c^-,B_2=c^+.$$

Now it is convenient to consider the directions $v(x,t)$ which satisfy

$$v(c,t)=v(A_i,t)=\frac{\partial v(A_i,t)}{\partial x}=\frac{\partial v(B_i,t)}{\partial x}=0, i=1,2,\forall t\in(t_a,t_b). \tag{8}$$

Using (8) in (7) and applying the stationary condition required by Hamilton´s principle (2), leads to

$$\delta F(u;v)=\int_{t_a}^{t_b}\sum_{i=1}^{2}\int_{\Omega_i}\left[m(x)\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right)-\right.$$
$$\left.-\frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right)\right]v(x,t)dxdt=0,\forall v\in D_a. \tag{9}$$

Let us assume $t_a=0$, then as $v(x,t)$ is an arbitrary smooth function, the fundamental lemma of the calculus of variations can be applied to Eq. (9) to conclude that the function $u(x,t)$ must satisfy the following differential equations:

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right)+m(x)\frac{\partial^2 u(x,t)}{\partial t^2}=0, \forall x\in\Omega_i, i=1,2, t\geq 0.$$

Now it is possible to remove the restrictions (8), and since the function $u(x,t)$ must satisfy the differential equations stated above, the expression (7) is reduced to

$$\delta F(u;v)=-\int_0^{t_b}\sum_{i=1}^{2}\left(P_i\frac{\partial v(A_i,t)}{\partial x}+Q_i v(A_i,t)+R_i\frac{\partial v(B_i,t)}{\partial x}+Z_i v(c,t)\right)dt. \tag{10}$$

Since, the functions $\frac{\partial v(A_i,t)}{\partial x}, v(A_i,t), \frac{\partial v(B_i,t)}{\partial x}$ and $v(c,t)$ are smooth and arbitrary, the stationary condition (2) applied to (10) leads to the boundary and transitions conditions. For instance, if we adopt

$$v(0^+,t)=\frac{\partial v(c^-,t)}{\partial x}=\frac{\partial v(c^+,t)}{\partial x}=v(c,t)=\frac{\partial v(l^-,t)}{\partial x}=v(l^-,t)=0, \forall t\in(0,t_b),$$

we obtain $$r_1\frac{\partial u(0^+,t)}{\partial x}=D(0^+)\frac{\partial^2 u(0^+,t)}{\partial x^2}.$$

In an analogue form all the rest of the boundary conditions and transitions conditions are obtained. It has been demonstrated that the function $u(x,t)$ must satisfy the boundary and eigenvalue problem shown in Table 2.

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right)+m(x)\frac{\partial^2 u(x,t)}{\partial t^2}=0,\ \forall x\in(0,c), \tag{11}$$

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right)+m(x)\frac{\partial^2 u(x,t)}{\partial t^2}=0, \forall x\in(c,l), \tag{12}$$

$$r_1\frac{\partial u(0^+,t)}{\partial x}=D(0^+)\frac{\partial^2 u(0^+,t)}{\partial x^2}, \tag{13}$$

$$t_1 u(0^+,t)=-\frac{\partial}{\partial x}\left(D(0^+)\frac{\partial^2 u(0^+,t)}{\partial x^2}\right)+T(0^+)\frac{\partial u(0^+,t)}{\partial x}, \tag{14}$$

$$r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}, \tag{15}$$

$$r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)-r_c\frac{\partial u(c^-,t)}{\partial x}=D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}, \tag{16}$$

$$t_c u(c,t)=\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right)-T(c^-)\frac{\partial u(c^-,t)}{\partial x}+T(c^+)\frac{\partial u(c^+,t)}{\partial x}, \tag{17}$$

$$r_2\frac{\partial u(l^-,t)}{\partial x}=-D(l^-)\frac{\partial^2 u(l^-,t)}{\partial x^2}, \tag{18}$$

$$t_2 u(l^-,t)=\frac{\partial}{\partial x}\left(D(l^-)\frac{\partial^2 u(l^-,t)}{\partial x^2}\right)-T(l^-)\frac{\partial u(l^-,t)}{\partial x}, \quad \text{where } t\geq 0. \tag{19}$$

Table 2. Boundary and eigenvalue problem.

2.2 The transition conditions.

Since the domain of definitions of the problem is $\Omega = (0,l)$ and this is an open interval in $\mathbb{R}$, the boundary is given by two points, i.e. $\partial\Omega = \{0,l\}$. Consequently, only the Eqs. (13), (14), (18) and (19) correspond to the boundary conditions. The point $x = c$ is an interior point of Ω, and the equations formulated at $x = c^-$ and $x = c^+$ can be called transition conditions. Consequently, the Eqs. (15), (16) and (17) correspond to the transition conditions of the problem. Since $u(\bullet,t) \in C(\overline{\Omega})$, there exists continuity of deflection at the point $x = c$ and this generate the transition condition $u(c^-,t) = u(c^+,t) = u(c,t)$.

2.3 The weak solution.

Variational methods were extensively used by engineers and scientists as a very effective tool for the solution of boundary and/or eigenvalue problems. At the same time several problems emerged of both theoretical and practical character. The finite element method is the most widely used technique for engineering design and analysis. This method provides a formalism for generating finite algorithms for approximating the solutions of boundary and/or eigenvalue problems. It works as a black box in which one puts the boundary and/or eigenvalue problem data and out of which is generated an algorithm for approximating the corresponding solutions. A part of this task can be done automatically by a computer, but it is necessary an amount of mathematical skill. It proved that to find an answer to a number of questions which are theoretically interesting and practically urgent is not a simple task. The functional analysis plays an essential role in the solutions of these problems and particularly the theory of Sobolev spaces and the concept of weak solution.

- **The statical case.**

The classical solution of the boundary and eigenvalue problem presented in Table 2, when there is no restrictions or hinge at the intermediate point c, is a function $u(x,t)$ such that $u(x,\bullet) \in C^2[t_a,t_b]$, $u(\bullet,t) \in C^4(0,l)$ and $u(\bullet,t) \in C^2[0,l]$. In other words, $u(\bullet,t)$ must have fourth-order partial derivatives continuous in the open interval $\Omega = (0,l)$, (since it must satisfy the differential equation) and two-order derivatives and function values continuous in the close interval $\overline{\Omega} = [0,l]$, since it must satisfy the boundary conditions which involve the extremes of this interval. But when there exist an internal hinge and elastic restraints in c, as it has been shown in Section 2, the function $u(x,t)$ does not have derivatives $\dfrac{\partial^n u(x,t)}{\partial x^n}$, with $n \geq 2$ in the interval Ω. In consequence, the boundary and eigenvalue problem presented in Table 2, does not have a classical solution. So, it is necessary to analyse the existence of a weak solution. Let us consider the statical behaviour of the mechanical system described, when a load $q = q(x)$, which causes a transverse deflection $w(x)$, is applied.

It is governed by the corresponding boundary value problem presented in Table 3, which was obtained with an analogue procedure to that used in section 2.1

$$\frac{d^2}{dx^2}\left(D(x)\frac{d^2w(x)}{dx^2}\right)-\frac{d}{dx}\left(T(x)\frac{dw(x)}{dx}\right)=q_1(x),\forall x\in\Omega_1, \tag{20}$$

$$\frac{d^2}{dx^2}\left(D(x)\frac{d^2w(x)}{dx^2}\right)-\frac{d}{dx}\left(T(x)\frac{dw(x)}{dx}\right)=q_2(x),\forall x\in\Omega_2, \tag{21}$$

$$r_1\frac{dw(0^+)}{dx}=D(0^+)\frac{d^2w(0^+)}{dx^2}, \tag{22}$$

$$t_1w(0^+)=-\frac{d}{dx}\left(D(0^+)\frac{d^2w(0^+)}{dx^2}\right)+T(0^+)\frac{dw(0^+)}{dx}, \tag{23}$$

$$r_{12}\left(\frac{dw(c^+)}{dx}-\frac{dw(c^-)}{dx}\right)=D(c^+)\frac{d^2w(c^+)}{dx^2}, \tag{24}$$

$$r_{12}\left(\frac{dw(c^+)}{dx}-\frac{dw(c^-)}{dx}\right)-r_c\frac{dw(c^-)}{dx}=D(c^-)\frac{d^2w(c^-)}{dx^2}, \tag{25}$$

$$t_cw(c)=\frac{d}{dx}\left(D(c^-)\frac{d^2w(c^-)}{dx^2}\right)-\frac{d}{dx}\left(D(c^+)\frac{d^2w(c^+)}{dx^2}\right)-T(c^-)\frac{dw(c^-)}{dx}+T(c^+)\frac{dw(c^+)}{dx}. \tag{26}$$

$$r_2\frac{dw(l^-)}{dx}=-D(l^-)\frac{d^2w(l^-)}{dx^2}, \tag{27}$$

$$t_2w(l^-)=\frac{d}{dx}\left(D(l^-)\frac{d^2w(l^-)}{dx^2}\right)-T(l^-)\frac{dw(l^-)}{dx}. \tag{28}$$

Table 3. Boundary value problem $\Omega_1=(0,c),\Omega_2=(c,l)$.

Let $H^2(\Omega)$ be the Sobolev space $H^2(\Omega)=\left\{u\in L^2(\Omega);D^\alpha u\in L^2(\Omega),\alpha=1,2\right\}$, where $\Omega=(0,l)$. This space can be equipped with the norm $\left\|u\right\|_{H^2(\Omega)}=\left(\sum_{\alpha=0}^{2}\int_\Omega(D^\alpha u)^2\,dx\right)^{\frac{1}{2}}$, where $D^\alpha u=u^{(\alpha)}$ is the weak derivative of order α of the function u.

The stable and unstable boundary and transition conditions are of different nature so in order to clearly distinguish them, it is useful to introduce the space V, of elements of the Sobolev space $H^2(\Omega)$, which satisfy the corresponding stable homogeneous boundary and transition conditions. For instante, if we let $r_1,r_c,r_{12},t_1,t_c\to\infty$, in Eqs. (22)-(26), these conditions are reduced to $w(0^+)=\frac{dw(0^+)}{dx}=w(c)=\frac{dw(c^-)}{dx}=0$. Consequently, since a weak solution of the boundary value problem (20) and (22)-(28) is a function from the Sobolev space $H^2(\Omega_1)$, the space V_1 is given by

$$V_1=\left\{v_1;v_1\in H^2(\Omega_1),v_1(0^+)=\frac{dv_1(0^+)}{dx}=v_1(c)=\frac{dv_1(c^-)}{dx}=0\right\}. \tag{29}$$

Similarly, adopting $r_2,r_c,r_{12},t_c,t_2\to\infty$, we conclude that a weak solution of boundary value problem (21) and (24)-(28) is a function from the Sobolev space $H^2(\Omega_2)$, and the space V_2 is given by

$$V_2=\left\{v_2;v_2\in H^2(\Omega_2),v_2(c)=\frac{dv_2(c^+)}{dx}=v_2(l^-)=\frac{dv_2(l^-)}{dx}=0\right\}. \tag{30}$$

If r_c, r_{12}, t_c take finite values, the transition conditions at the point $x = c$ are unstable, so they do not belong to the spaces V_i. Moreover, when also the coefficients r_1, r_2, t_1, t_2 take finite values, there are no stable boundary conditions and the spaces V_i can be taken as $V_i = \{v_i; v_i \in H^2(\Omega_i)\}, i = 1,2.$

Let $q_i(x) \in C(\overline{\Omega}_i)$, $D_i(x) \in C^2(\overline{\Omega}_i)$, $T_i(x) \in C^1(\overline{\Omega}_i)$ and $w_i = w(\bullet, t)\big|_{\overline{\Omega}_i} \in C^4(\overline{\Omega}_i)$, be the classical solutions for the problem (20)-(28). Now this boundary value problem is transformed into one that leads to the concept of weak solution. If we take arbitrary functions $v_i \in V_i$, and multiply the Eqs. (20) and (21) respectively by these functions and integrate each result over the corresponding domain we get

$$\int_{\Omega_i} \frac{d^2}{dx^2}\left(D(x)\frac{d^2 w_i(x)}{dx^2}\right)v_i(x)dx - \int_{\Omega_i} \frac{d}{dx}\left(T(x)\frac{dw_i(x)}{dx}\right)v_i(x)dx = \int_{\Omega_i} q_i(x)v_i(x)dx, \quad i = 1,2.$$

Integrating by parts the two first integrals we obtain

$$\begin{aligned}
&\int_{\Omega_i} D(x)\frac{d^2 w_i(x)}{dx^2}\frac{d^2 v_i(x)}{dx^2}dx + \int_{\Omega_i} T(x)\frac{dw_i(x)}{dx}\frac{dv_i(x)}{dx}dx + \\
&+ \frac{d}{dx}\left(D(x)\frac{d^2 w_i(x)}{dx^2}\right)v_i(x)\bigg|_{\Omega_i} - D(x)\frac{d^2 w_i(x)}{dx^2}\frac{dv_i(x)}{dx}\bigg|_{\Omega_i} - \\
&- T(x)\frac{dw_i(x)}{dx}v_i(x)\bigg|_{\Omega_i} = \int_{\Omega_i} q_i(x)v_i(x)dx, \ \forall v_i \in V_i, i = 1,2.
\end{aligned} \tag{31-a,b}$$

Summing Eqs. (31-a) and (31-b) and taking into account the boundary and transition conditions (22)-(28) we obtain

$$\begin{aligned}
B(w,v) = &\int_{\Omega_1} D(x)\frac{d^2 w(x)}{dx^2}\frac{d^2 v(x)}{dx^2}dx + \int_{\Omega_2} D(x)\frac{d^2 w(x)}{dx^2}\frac{d^2 v(x)}{dx^2}dx + \\
&+ \int_{\Omega} T(x)\frac{dw(x)}{dx}\frac{dv(x)}{dx}dx + \sum_{i=1}^{3}\left(a_i\frac{dw(b_i)}{dx}\frac{dv(b_i)}{dx} + c_i w(d_i)v(d_i)\right) + \\
&+ r_{12}\left(\frac{dw(d_2)}{dx} - \frac{dw(b_2)}{dx}\right)\left(\frac{dv(d_2)}{dx} - \frac{dv(b_2)}{dx}\right) = \int_{\Omega} q(x)v(x)dx, \ \forall v \in V,
\end{aligned} \tag{32}$$

where

$$w(\bullet, t)\big|_{\Omega_i} = w_i(x), \ i = 1,2, \ q(x) = \begin{cases} q_1(x), \ \forall x \in [0,c), \\ q_2(x), \ \forall x \in (c,l]. \end{cases}$$

The coefficients a_i, b_i, c_i, d_i are the same defined in Table 1. Finally, the space V is given by $V = \left\{v; v \in H^1(\Omega), v\big|_{\Omega_i} \in H^2(\Omega_i), i = 1,2\right\}.$

It must be noted that if a function $u(x) \in H^2(\Omega)$ it implies that $u(x)$ and $u'(x)$ are continuous in $\Omega \subset \mathbb{R}$. But as stated above, the presence of the internal hinge implies $\dfrac{dw(c^+)}{dx} \neq \dfrac{dw(c^-)}{dx}$. In consequence $w \in H^1(\Omega)$. More precisely $w \in H^1(\Omega)$, and $w\big|_{\Omega_i} \in H^2(\Omega_i), i = 1,2.$ This is the reason for adopting the space V defined above.

The first two terms on the left hand side of (32) constitute the bilinear form $A(w,v)$ associated with the differential Eqs. (20) and (21). The other terms which are related to the boundary and transition conditions, correspond to the bilinear form $a(w,v)$. The equality (32) now assumes the form

$$B(w,v) = A(w,v) + a(w,v) = \int_\Omega qv dx = (q,v)_{L^2(\Omega)}, \forall v \in V.$$

Now we are going to weaken the assumptions. Let $q(x) \in L^2(\Omega), D(x) \in L^\infty(\Omega)$, $T(x) \in L^\infty(\Omega)$. A function w is called a weak solution of the boundary value problem (20)-(28) if

$$(i)\quad w \in V. \qquad (ii)\ B(w,v) = (q,v)_{L^2(\Omega)}, \forall v \in V. \tag{33)- (34}$$

It must be noted that in Eq. (32) $\frac{dw(x)}{dx}$ denotes the first order weak derivative of the function w in $\Omega = (0,l)$ but $\frac{d^2w(x)}{dx^2}$ is not necessarily a weak derivative, since a step function is not in the space $H^1(0,l)$. For this reason the expressions $\int_{\Omega_i} D(x)\frac{d^2w(x)}{dx^2}\frac{d^2v(x)}{dx^2}dx, i = 1,2$ are used, instead of the integral over Ω.

If the bilinear form $B(w,v)$ is continuous in V and $V-$elliptic, the problem under consideration has exactly one weak solution w, Necas (1967); Rektorys(1980). If we replace $w = v$, in Eq. (32) we obtain

$$\begin{aligned} B(w,w) - 2(q,w)_{L^2(\Omega)} &= \int_{\Omega_1} D(x)\left(\frac{d^2w(x)}{dx^2}\right)^2 dx + \int_{\Omega_2} D(x)\left(\frac{d^2w(x)}{dx^2}\right)^2 dx + \\ &+\int_\Omega T(x)\left(\frac{dw(x)}{dx}\right)^2 dx + \sum_{i=1}^{3}\left(a_i\left(\frac{dw(b_i)}{dx}\right)^2 + c_i w^2(d_i)\right) + \\ &r_{12}\left(\frac{dw(c^+)}{dx} - \frac{dw(c^-)}{dx}\right)^2 - 2\int_\Omega q(x)w(x)dx. \end{aligned} \tag{35}$$

We can recognize that (35) is proportional to the potential energy of the system under study. Since the bilinear form $B(w,v)$ is also symmetric, the function $w(x)$ is the weak solution of the problem (20)-(28), if and only if it minimizes, in the space V, the functional, Necas (1967); Rektorys(1980)

$$I(w) = \frac{1}{2}B(w,w) - (w,q)_{L^2(\Omega)}, \forall v \in V. \tag{36}$$

The Ritz method can be applied adopting the approximating function

$$w_N(x) = \sum_{i=1}^{N} c_{Ni}\varphi_i(x),$$

where $\varphi_i(x)$ are elements of a base in V. The coefficients c_{Ni} are determined by the condition $I(w_N) = \min$. This procedure leads to the following system of linear equations,

$$\sum_{j=1}^{N} c_{Nj} B(\varphi_i, \varphi_j) = (\varphi_i, q)_{L^2(\Omega)}, \quad i = 1, 2, \ldots, N. \tag{37}$$

• **The eigenvalue problem.**

In the case of normal modes of vibrations we take $u(x,t) = w(x)\cos\omega t$, where ω is the natural radian frequency. Consequently Eqs. (11)-(19), are reduced to

$$\frac{d^2}{dx^2}\left(D(x)\frac{d^2 w(x)}{dx^2}\right) - \frac{d}{dx}\left(T(x)\frac{dw(x)}{dx}\right) - \omega^2 m(x) w(x) = 0, \forall x \in \Omega_1, \tag{38}$$

$$\frac{d^2}{dx^2}\left(D(x)\frac{d^2 w(x)}{dx^2}\right) - \frac{d}{dx}\left(T(x)\frac{dw(x)}{dx}\right) - \omega^2 m(x) w(x) = 0, \forall x \in \Omega_2, \tag{39}$$

and the conditions (22)-(28).
In this case the problem of finding a number $\bar{\lambda}$ and a function w such that

$$\begin{cases} w \in V, \ w \neq 0, \\ B(w,v) - \bar{\lambda}(w,v) = 0, \quad \forall v \in V, \end{cases} \tag{40}$$

where $\bar{\lambda} = \rho(0)A(0)\omega^2$, $(w,v) = \int_\Omega h(x)w(x)v(x)dx$, is the eigenvalue problem of the bilinear form $B(w,v)$.
If it is symmetric, continuous and $V-$elliptic, then it has a countable set of eigenvalues and are given by, Necas (1967); Rektorys(1980):

$$\bar{\lambda}_1 = \min\left\{\tfrac{B(v,v)}{(v,v)}, \ v \in V, v \neq 0\right\},$$

$$\bar{\lambda}_N = \min\left\{\tfrac{B(v,v)}{(v,v)} \ v \in V, v \neq 0, \ (v, v_1) = ... = (v, v_N) = 0\right\}.$$

Let us introduce a new inner product in space V given by $((w,v)) = B(w,v), \forall v \in V$. If the sequence $\{\varphi_i(x)\}_{i=1}^{\infty}$ is a base in the space V with the inner product $((w,v))$, the Ritz method leads to the equation

$$\begin{vmatrix} ((\varphi_1,\varphi_1)) - \bar{\lambda}(\varphi_1,\varphi_1) & \cdots & ((\varphi_1,\varphi_N)) - \bar{\lambda}(\varphi_1,\varphi_N) \\ & & \\ ((\varphi_N,\varphi_1)) - \bar{\lambda}(\varphi_N,\varphi_1) & \cdots & ((\varphi_N,\varphi_N)) - \bar{\lambda}(\varphi_N,\varphi_N) \end{vmatrix} = 0. \tag{41}$$

The approximate eigenvalues can be obtained from (41), when dealing with the dynamical behaviour of the beam considered above.

3. ANISOTROPIC PLATES.

Let us consider an anisotropic plate that in the equilibrium position covers the two-dimensional domain Ω, with smooth boundary $\partial\Omega$ elastically restrained against rotation and translation. The plate is also restrained against translation along an intermediate line and has an internal hinge elastically restrained against rotation, as it is shown in Figure 2.

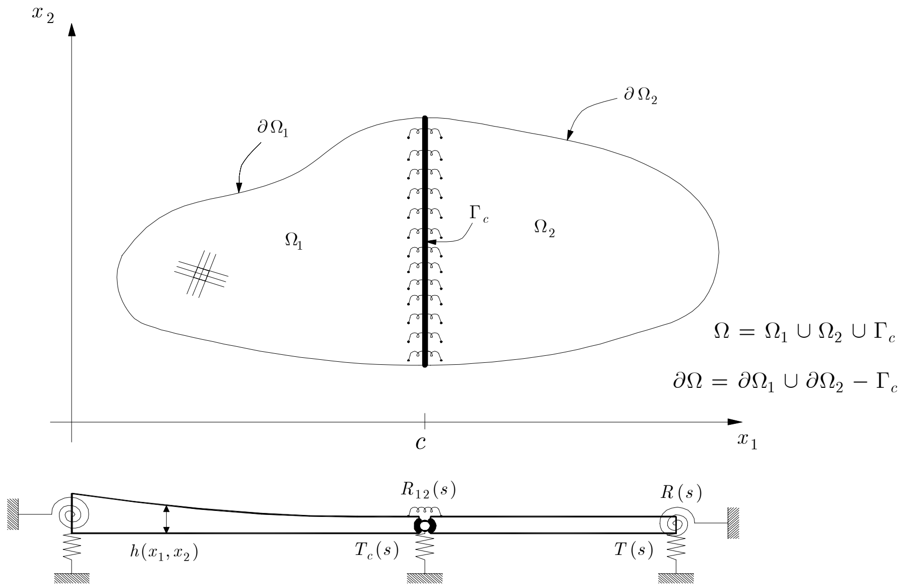


Figure 2. Mechanical system under study.

This study is restricted to thin flat plates of mid-plane symmetry. In order to analyze the transverse displacements of the system under study we suppose that the vertical position of the plate at any time t, is described by the function $w = w(x_1, x_2, t)$, where $(x_1, x_2) \in \overline{\Omega}$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. It is also assumed that the plate is divided in two domains Ω_1 and Ω_2 with boundaries $\partial\Omega_1$ and $\partial\Omega_2$ to which respectively correspond different rigidities $D_{kl}(x_1, x_2)$ and mass density $m_i(x_1, x_2) = \rho_i(x_1, x_2)h_i(x_1, x_2)$, of the anisotropic material. The rotational restraints are characterized by the spring constants $R\left(s\right)$ and $R_{12}\left(s\right)$ and the translational restraints by the spring constants $T\left(s\right)$ and $T_c\left(s\right)$ where s is the arc length along the boundary $\partial\Omega$ and the line Γ_c.

At time t, the kinetic energy of the plate is given by

$$T(w) = \frac{1}{2}\iint_{\Omega_1} \rho_1 h_1 \left(\frac{\partial w}{\partial t}\right)^2 dx_1 dx_2 + \frac{1}{2}\iint_{\Omega_2} \rho_2 h_2 \left(\frac{\partial w}{\partial t}\right)^2 dx_1 dx_2.$$

On the other hand, at time t, the total potential energy due to the elastic deformation of the plate deformed by a load of density $q = q\left(x_1, x_2, t\right)$ acting on Ω, to the elastic restraints on the boundary $\partial\Omega$, to the elastic restraints at the intermediate line and at the internal hinge is given by:

$$U(w) = \frac{1}{2}\sum_{i=1}^{2}\left\{\iint_{\Omega_i}\left[D_{11}^{(i)}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + D_{22}^{(i)}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2D_{12}^{(i)}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + 4D_{66}^{(i)}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 + \right.\right.$$

$$\left. + 4\frac{\partial^2 w}{\partial x_1 \partial x_2}\left(D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + D_{26}^{(i)}\frac{\partial^2 w}{\partial x_2^2}\right) - 2qw\right]dx_1 dx_2 + \int_{\Gamma_i} R(s)\left(\frac{\partial w}{\partial n^{(i)}}\right)^2 ds +$$

$$\left. + \int_{\Gamma_i} T(s) w^2 ds\right\} + \frac{1}{2}\int_{\Gamma_c} T_c(s) w^2 ds + \frac{1}{2}\int_{\Gamma_c} R_{12}(s)\left(\frac{\partial w\left(c^+, x_2, t\right)}{\partial x_1} - \frac{\partial w\left(c^-, x_2, t\right)}{\partial x_1}\right)^2 ds,$$

where $\Gamma_i = \partial\Omega_i - \Gamma_c$. The line $\Gamma_c = \{(c, x_2) \in \Omega\}$ is the common part of the boundaries $\partial\Omega_1$, $\partial\Omega_2$, and $\dfrac{\partial w}{\partial n^{(i)}}$ is the derivative of w with respect to the corresponding outward normal where $n_1^{(i)}, n_2^{(i)}$ are the components of the exterior unit normal on $\partial\Omega_i$.

The notations c^- and c^+ imply the use of lateral partial derivatives. It can be observed that the strain energy due to the rotational restraint of the internal hinge, is computed by $\dfrac{1}{2}\int_{\Gamma_c} R_{12}(s)\left(\dfrac{\partial w(c^+, x_2, t)}{\partial x_1} - \dfrac{\partial^2 w(c^-, x_2, t)}{\partial x_2^2}\right)^2 ds$ which implies that the distributed spring is connected at points in Ω_1 and at points in Ω_2.

Hamilton´s principle requires that between times t_0 and t_1, at which the positions of the mechanical system are known, it should execute a motion which makes stationary the functional $F(w) = \int_{t_0}^{t_1} (T - U)\, dt$, on the space of admissible functions. In consequence the energy functional to be considered is given by

$$
\begin{aligned}
F(w) = \frac{1}{2}\int_{t_0}^{t_1} \Bigg\{ \sum_{i=1}^{2} \Bigg[\int_{\Omega_i} \Bigg(m\left(\frac{\partial w}{\partial t}\right)^2 - D_{11}^{(i)}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 - 2D_{12}^{(i)}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - D_{22}^{(i)}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 - \\
-4\frac{\partial^2 w}{\partial x_1 \partial x_2}\left(D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + D_{26}^{(i)}\frac{\partial^2 w}{\partial x_2^2}\right) - 4D_{66}^{(i)}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 + 2qw \Bigg) dx - \int_{\Gamma_i} R^{(i)}(s)\left(\frac{\partial w}{\partial n^{(i)}}\right)^2 ds \qquad (42) \\
- \int_{\Gamma_i} T^{(i)}(s) w^2 ds \Bigg] - \frac{1}{2}\int_{\Gamma_c} T_c(s) w^2 ds - \frac{1}{2}\int_{\Gamma_c} R_{12}(s)\left[\frac{\partial w(c^+, x_2, t)}{\partial x_1} - \frac{\partial w(c^-, x_2, t)}{\partial x_1}\right]^2 ds \Bigg\} dt,
\end{aligned}
$$

where $x = (x_1, x_2)$, $w = w(x, t)$,

$$
q(x,t) = \begin{cases} q_1(x,t), \ \forall x \in \Omega_1 \\ q_2(x,t), \ \forall x \in \Omega_2 \end{cases}, \quad m(x) = \begin{cases} m_1(x) = \rho_1(x) h_1(x), \ \forall x \in \Omega_1, \\ m_2(x) = \rho_2(x) h_2(x), \ \forall x \in \Omega_2, \end{cases}
$$

The stationary condition for the functional (42) requires that

$$
\delta F(w, v) = 0, \forall v \in D_a \qquad (43)
$$

where $\delta F(u, v)$ is the first variation of F at u in the direction v and D_a is the space of admissible directions at w for the domain D of this functional. The application of the techniques of the calculus of variations leads to:

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_1^2}\left(D_{11}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + D_{12}^{(i)}\frac{\partial^2 w}{\partial x_2^2} + 2D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right) + \\
&+\frac{\partial^2}{\partial x_2^2}\left(D_{12}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + D_{22}^{(i)}\frac{\partial^2 w}{\partial x_2^2} + 2D_{26}^{(i)}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right) + \\
&+\frac{\partial^2}{\partial x_1 \partial x_2}\left(2D_{16}^{(i)}\frac{\partial^2 w}{\partial x_1^2} + 2D_{26}^{(i)}\frac{\partial^2 w}{\partial x_2^2} + 4D_{66}^{(i)}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right) + \\
&+m_i\frac{\partial^2 w}{\partial t^2} - q_i = 0, \ \forall x \in \Omega_i, i = 1,2, \ \forall t \geq 0.
\end{aligned} \qquad \text{(44 a,b)}
$$

$$R(s)\frac{\partial w}{\partial n}\bigg|_{\partial\Omega} = M|_{\partial\Omega}, \qquad T(s)w|_{\partial\Omega} = N - \frac{\partial P}{\partial s}\bigg|_{\partial\Omega}, \tag{45) - (46}$$

$$\begin{aligned} T_c(s)w(c^-,x_2,t) = -\left(N_1(c^-,x_2,t) + \frac{\partial H_{12}}{\partial x_2}(c^-,x_2,t)\right) + \\ +\left(N_1(c^+,x_2,t) + \frac{\partial H_{12}}{\partial x_2}(c^+,x_2,t)\right), \end{aligned} \tag{47}$$

$$R_{12}(s)\left(\frac{\partial w(c^+,x_2,t)}{\partial x_1} - \frac{\partial w(c^-,x_2,t)}{\partial x_1}\right) = -M_1(c^-,x_2,t) = -M_1(c^+,x_2,t), \tag{48}$$

4. NATURAL FREQUENCIES AND MODE SHAPES.

In order to check the accuracy of the algorithms developed, the frequency parameters were computed for a number of beams problems for which comparison values were available in the pertinent literature. Additionally, a great number of problems were solved and since the number of cases was extremely large, results were selected for the most significant cases. The analytical expressions obtained allow the adoption of different values for the following parameters:

- mass per unit length and flexural rigidity, of the ith span,
- rotational and translational restraint coefficients,
- axial forces T_1, T_c and T_2,
- distributed force $f(x)$,
- position of point *c*.

Using the well-known method of separation of variables, when the mass per unit length, the flexural rigidity and the axial force at the ith span are constant, we assume as solutions of Eqs. (11) and (12) respectively the expressions

$$u_1(x,t) = \sum_{n=1}^{\infty} u_{1,n}(x)\cos\omega t, \qquad u_2(x,t) = \sum_{n=1}^{\infty} u_{2,n}(x)\cos\omega t, \tag{49) – (50}$$

where $u_{1,n}(x)$ and $u_{2,n}(x)$ are the corresponding nth modes of natural vibration.

Introducing the change of variable $\overline{x} = x/l$, in Eqs. (11)-(19) the functions $u_{1,n}(x)$ and $u_{2,n}(x)$ are given by

$$u_{1,n}(\overline{x}) = A_1\cosh a_1\overline{x} + A_2\sinh a_1\overline{x} + A_3\cos b_1\overline{x} + A_4\sin b_1\overline{x}, \tag{51}$$

$$u_{2,n}(\overline{x}) = A_5\cosh a_2\overline{x} + A_6\sinh a_2\overline{x} + A_7\cos b_2\overline{x} + A_8\sin b_2\overline{x} \tag{52}$$

and the following dimensionless parameters can be defined:

$$a_i = \sqrt{\frac{S_i}{2} + \Delta_i}, b_i = \sqrt{-\frac{S_i}{2} + \Delta_i}, \Delta_i = \sqrt{\frac{S_i^2}{4} + \lambda_i^4}, \; S_i = \frac{T_i l^2}{D_i}, \; \lambda_i^4 = \frac{\omega^2 m_i}{D_i} l^4, i = 1,2,$$

Substituting Eqs. (51) and (52) in Eqs. (49) and (50) and then in the boundary conditions (13),(14),(18),(19) and transition conditions (15)-(17) we obtain a set of eight homogeneous equations in the constants A_i. Since the system is homogeneous for existence of a non trivial solution the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G(K_{ri}, K_{ti}, K_{rc}, K_{r12}, K_{tc}, S_i, \lambda_1, c) = 0 \tag{59}$$

where

$$K_{ri} = \frac{r_i l}{D_i}, K_{ti} = \frac{t_i l^3}{D_i}, i = 1,2, \qquad K_{rc} = \frac{r_c l}{D_1}, K_{r12} = \frac{r_{12} l}{D_1}, K_{tc} = \frac{t_c l^3}{D_1}. \tag{60}$$

The values of the frequency parameter $\lambda_1 = \left(\omega^2 m_1 / D_1\right)^{1/4} l,$ were obtained with the classical bisection method and rounded to six decimal digits.

Table 4 depicts the first three exact values of the frequency parameter λ_1 of a uniform beam with and free internal hinge. Two different boundary conditions and values of S_1 and S_2 are considered. A comparison of values of the fundamental frequency with those of Wang and Wang (2001) given in plots, shows an excellent agreement from an engineering viewpoint.

		$C-C$			$C-SS$		
$S_1 = S_2$	$\frac{c}{l}$	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$
20	0.5	4.803415	8.289007	10.080855	4.332703	7.492574	9.573966
	0.4	4.884738	7.836494	11.003137	4.455115	7.012395	10.567871
	0.3	5.099945	7.439040	11.009806	4.605980	6.853667	10.111099
	0.2	5.218508	7.856133	10.321988	4.621299	7.367559	9.649816
	0.1	5.001689	8.225792	11.338858	4.425578	7.514569	10.625665
	0	4.642849	7.609618	10.625710	4.143643	6.961131	9.915604
50	0.5	5.562708	8.831742	10.881771	5.172028	8.197910	10.392662
	0.4	5.598529	8.549851	11.566439	5.217433	7.904708	11.116684
	0.3	5.692301	8.288043	11.570925	5.283389	7.786504	10.800847
	0.2	5.759859	8.542097	11.034230	5.310188	8.074734	10.447595
	0.1	5.636741	8.795938	11.810324	5.196631	8.197825	11.167929
	0	5.312890	8.249932	11.169513	4.930336	7.709381	10.537889

Table 4. First three exact values of the frequency parameter λ_1, of a uniform beam with a free internal hinge, two boundary conditions and two different values of S_1 and S_2. (C : clamped, SS : simply supported)

Table 5 depicts the first three exact values of the frequency parameter λ_1 of a uniform beam with an free internal hinge and different boundary conditions. The mode shapes which correspond to a hinge located at $c/l = 0.5$ are also presented.

Table 6 depicts exact values of the fundamental frequency parameter $\lambda_{1,1}$, of a uniform beam clamped at $\bar{x} = 0,$ with an intermediate point elastically restrained against rotation and translation, with an elastically restrained internal hinge and free at $\bar{x} = 1.$ When $K_{r12} \to \infty$ the values obtained agree with those of reference Grossi and Albarracín (2003).

In Figure 3 the exact fundamental frequency parameter $\lambda_{1,1}$ is plotted against the restraint parameters K_{r2} and K_{t2}. The beam is clamped at $\bar{x} = 0$ and elastically restrained at $\bar{x} = 1.$ The free internal hinge $(K_{r12} = K_{rc} = K_{tc} = 0)$ is located at two different points $c/l.$ It can be observed that major increase of frequency occur when the elastic restrain values are in the interval $[10, 1000]$.

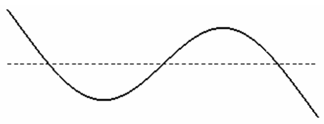
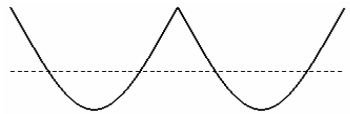
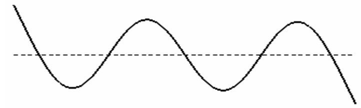
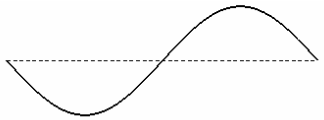
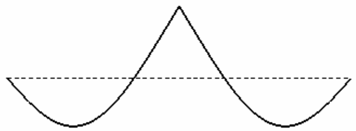
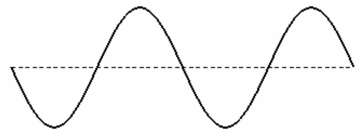
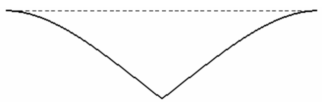
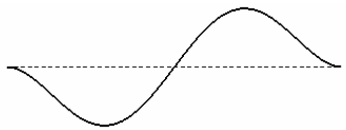
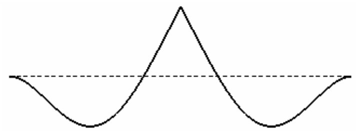
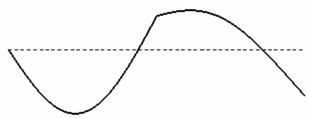
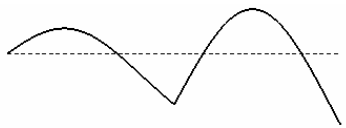
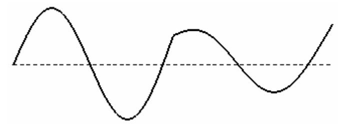
B.C.	c/l	$\lambda_{1,1}$	Mode Shape	$\lambda_{1,2}$	Mode Shape	$\lambda_{1,3}$	Mode Shape
$F-F$	0.5	7.853205		9.460081		14.137165	
	0.4	7.149421		10.610666		12.733213	
	0.3	6.301510		10.528024		13.933756	
	0.2	5.639917		9.421523		13.235406	
	0.1	5.129561		8.532251		11.964969	
	0	4.730041		7.853205		10.995608	
$SS-SS$	0.5	6.283185		7.853205		12.566371	
	0.4	5.782606		8.767856		11.312915	
	0.3	5.131786		9.276929		11.780378	
	0.2	4.618322		8.391546		12.161731	
	0.1	4.226369		7.631299		11.050510	
	0	3.141593		6.283185		9.424778	
$C-C$	0.5	3.750208		7.853205		9.388182	
	0.4	3.953408		7.180653		10.560987	
	0.3	4.483116		6.584228		10.573598	
	0.2	4.706597		7.225821		9.737986	
	0.1	4.348263		7.757249		10.987577	
	0	3.926602		7.068583		10.210176	
	1	3.926602		7.068583		10.210176	
	0.9	4.256359		7.676480		11.106217	
	0.8	4.673940		8.469451		12.283192	
	0.7	5.214137		9.478488		13.342975	
	0.6	5.922674		10.167962		11.798847	
$SS-F$	0.5	6.786463		8.926649		13.090828	
	0.4	6.920416		9.052879		12.468217	
	0.3	6.205473		10.198446		12.124760	
	0.2	5.577540		9.336974		13.089156	
	0.1	5.094911		8.484132		11.907033	
	0	3.926602		7.068583		10.210176	

Table 5: First three exact values of the frequency parameter λ_1 of a uniform beam with an free internal hinge and different boundary conditions. The mode shapes which correspond to a hinge located at $c/l = 0.5$ are also presented.

c/l	$K_{tc} = K_{rc}$	K_{r12}						
		∞		1000	100	10	1	0
		Present	Grossi and Albarracín (2003)					
0.6	0	1.87510	1.87510	1.87500	1.87411	1.86527	1.78378	2.74615
	1	2.09119	2.09119	2.09102	2.08951	2.07456	1.94204	2.98916
	10	2.74618	2.74618	2.74558	2.74017	2.68723	2.29297	3.58787
	100	3.67938	3.67938	3.67670	3.65271	3.43269	2.49446	4.62004
	1000	4.53275	4.53275	4.52282	4.43794	3.86571	2.54981	6.84224
	10000	4.67264	4.67264	4.66114	4.56340	3.92741	2.55730	7.79768
0.4	0	1.87510	-	1.87471	1.87114	1.83709	1.60366	3.68304
	1	2.01557	-	2.01499	2.00986	1.96170	1.66141	3.90061
	10	2.50929	-	2.50755	2.49212	2.35917	1.79295	4.54802
	100	2.95500	-	2.95100	2.91623	2.65068	1.85214	5.26730
	1000	3.09608	-	3.09110	3.04789	2.73066	1.86491	6.27871
	10000	3.12195	-	3.11679	3.07205	2.74529	1.86719	6.51932

Table 6. Values of the fundamental frequency parameter $\lambda_{1,1}$, of a uniform cantilever beam with an intermediate point elastically restrained against rotation and traslation $(K_{tc} = K_{rc})$ and with an elastically restrained internal hinge located at two different points, $c/l = 0.4, 0.6$.

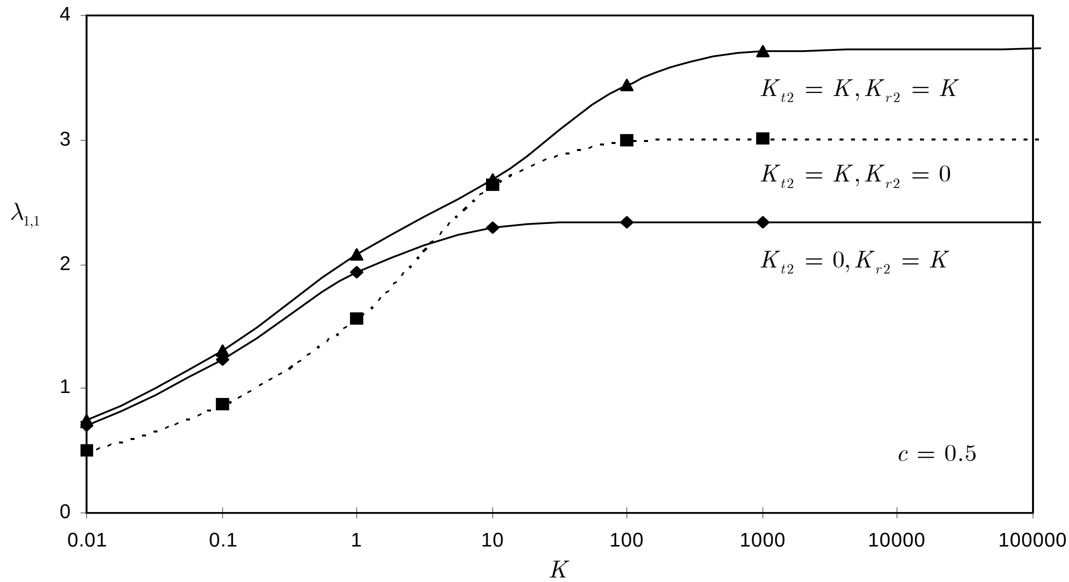
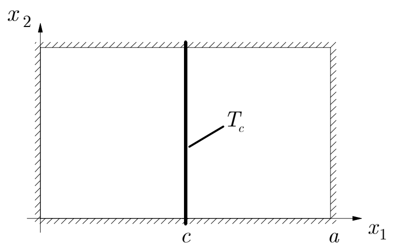


Figure 3: Variation of fundamental frequency parameter $\lambda_{1,1}$ with rotational and translational restraint parameters K_{r2} and K_{t2} of a clamped-elastically restrained beam with an free internal hinge $(K_{12} = K_{rc} = K_{tc} = 0)$ located at $c/l = 0.5$.

Table 7 depicts values of the first three frequency parameters $\Omega_i = \left(\omega_i^2 m / D_1\right)^{1/4} a,$ of a uniform square plate having all sides simply support with an intermediate line elastically restrained against translation. The values were obtained using the Ritz method with a polynomial base.



T_c	Ω_1	Ω_2	Ω_3
5000	49.3480	62.2918	78.9568
500	35.6646	49.3480	57.9093
50	22.1002	49.3480	50.3427
5	19.9906	49.3480	49.4491
0.5	19.7645	49.3480	49.3581
0.05	19.7417	49.3480	49.3490

Table 7: First three frequency parameters $\Omega_i = \left({\omega_i^2 m}/{D_1} \right)^{1/4} a,$ of a uniform square plate having all sides simply support with an intermediate line elastically restrained against translation.

5. CONCLUSIONS

A simple, computationally efficient and accurate approach has been developed for the determination of natural frequencies and modal shapes of free vibration of a non homogeneous tapered beam subjected to general axial forces, with arbitrarily located internal hinge and elastics supports, and ends elastically restrained against rotation and translation. Also the free transverse vibration of anisotropic plates of different geometrical, generally restrained boundaries which is restrained against translation along an intermediate line and has an internal hinge elastically restrained against rotation.

. Hamilton´s principle has been rigorously applied to obtain the differential equations, boundary conditions, and particularly the transitions conditions. The algorithms are very general and are attractive regarding its versatility in handling any boundary conditions and any transition conditions, including ends and an intermediate point elastically restrained against rotation and translation. Besides, these algorithms allow to take into account a great variety complicating effects such us: thickness variation, different types of axial forces and an arbitrarily located internal hinge with a rotational restraint. Close agreement with results presented by previous investigators is demonstrated for several examples. New results are presented for several beams with internal hinge and elastic restraints and a plate with an intermediate line elastically restrained. These results may provide useful information for structural designers and engineers.

It has been demonstrated that the boundary and the eigenvalue problem which respectively describe the statical and dynamical behaviour of the mechanical systems analysed, do not have classical solutions. The problem of existence and uniqueness of the weak solutions of the corresponding boundary value problems and eigenvalue problems has been treated.

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