AUCTION THEORY FOR SUDDEN-DEATH CHESS GAMES: OR BROADLY SPEAKING, TWO-GOOD, TWO-AGENT ASSIGNMENT PROBLEMS

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Keywords: Mechanism Design, Auction Theory, Multiple-good Auctions, Bayesian Nash Equilibrium, Sudden-Death Chess.

Abstract. This paper studies a family of bargaining problems. Although the first motivational example is based on a chess case (that of assigning sides and times in sudden-death games), the model is able to interpret any situation in which two goods must be allotted between two agents, allowing for compensations and payments.

Under a linear formulation, equilibria for a series of mechanisms are found. Studied mechanisms include the classical auctions. We also analyze *interim* properties of general assignment rules over this framework. The problem is translated to an outline equivalent to the traditional format of canonical auctions. Some of the most important results in terms of Mechanism Design comprise translations of orthodox principles found in the literature related to the customary bargaining problem of auctioneering one good among *n* potential buyers [under the Independent Private Valuations Model, *IPVM*]. Traditional auctions are again found to be efficient among the family of feasible rules. We also present an example of a factual (implementable in practice) direct revelation mechanism for the uniform case.

The model could be extended in at least three ways: First, asymmetries could be included. Second, the structure could be changed to include non linear formulations (for instance, risk aversion postulates). Last, but not least, comes the issue of extending the model for *n*-good, *n*-person assignment problems.

1 INTRODUCTION

The purpose of this paper is to analyze a framework in which two goods must be allotted between two individuals, and a means of transfer is provided to allow for payments and compensations.

Initially, a motivational case related to chess is presented, though all the results are valid for any situation for which the model can be applied.

1.1 Sudden death chess games

There are three possible outcomes to a chess game: a win by either player, or a draw. Many matches and other chess contests usually need for some kind of tie-breaking procedure if a winner is to be declared. Performing tie-breaking rapid mini-matches as a sequel to traditional, slow-time-control matches is a common method for picking a winner on a leveled finish. However, many times these mini-matches also end with a tie, so a second stage of tie-breaking procedure takes place.

Recently, a common second-stage tie-breaking procedure has appeared: the sudden-death [*Armageddon*] chess game, its main feature being that of always picking a final winner and eluding the need for a further stage of tiebreak.

The Armageddon game consists of a single game (usually with very fast [blitz] time controls, say 4-6 minutes in a player's clock), favoring White with more time on its clock, but granting Black draw odds, so White is obliged to win the game to prevail in the tiebreak. The most frequent array is to give the white side a fixed amount of time (for instance, 6 minutes) and Black some time less (usually 5 minutes against White's 6, or 4 minutes against White's 5). The right to choose side is decided by random.

In this scenery, auctions have occasionally taken place. As it will be shown in this work, an auction will produce a more efficient result than the fixed rule stated above. In general, those rules perform indeed better in terms of efficiency than any other rule that leaves space to random or fails to process an adequate amount of signals from the agents.

The main purpose of this paper is to analyze through a simple model the main features of the problem in terms of Mechanism Design theory, examining a series of possible alternatives for allotting times and sides.

1.2 The model

Basic structure

The frame of the model is the following: We suppose there is a fixed time assigned for the white side, say t^W, and the proposed mechanisms must assign sides and time for the black player.

There are two players, who perceive functions $P_i^W(t)$ and $P_i^B(t)$ ($i = 1,2$), which denote the probability that agent *i* respectively assigns to winning with the white pieces or not losing (i.e. winning or drawing) while playing on the black side, when time allotted to the black player is *t*. Arguably, $P_i^W(t)$ and $P_i^B(t)$ can be supposed [monotonically] decreasing and increasing, respectively.

If we assume that the sudden-death (tie-breaking) scheme applies to players who do not differ much in level of play, it is reasonable to suppose that there is a real number τ_i for which $P_i^W(\tau_i) = P_i^B(\tau_i)$.

Furthermore, we suppose that the *ex post* utility of agent *l* [$l \in \{1;2\}, -l = \{1;2\} \setminus l$] takes the form[1]

$$U_l\left(\tau_l, t, \delta_l^W\right) = \overline{U}_l + \delta_l^W\left(\tau_l - t\right) + \delta_{-l}^W\left(t - \tau_l\right) \qquad \textbf{[1]}$$

…where t is the time allotted to Black, and δ_l^W is the probability of player l getting the white pieces. Notice that this formulation implies symmetry and linearity of the effects on the difference of times.

We continue our assumptions conjecturing that the possible types of players to interact (assuming symmetry and *statistical* independence of players) can be characterized by a cumulative probability function $F(\cdot)$ with support given by the real interval $\left[\underline{\tau};\overline{\tau}\right]$. Furthermore, we suppose that $F(\cdot)$ is differentiable, hence, $f(\tau) = \frac{\mathrm{d}F(\tau)}{\mathrm{d}\tau}$ is the density probability function implied by $F(\tau)$. We also assume a model of *private types*, independent *in value*, in the sense that no agent l adjusts her figure τ_l, no matter which signals she is able to perceive (i. e. what information on τ_{-l} she is able to infer from those signals). In other words, we sustain that each player's determination of τ_l rests solely on her own chess knowledge and understanding of her rival's characteristics[2].

Given the form of $U_l(\cdot)$ defined in **[1]**, we can set (with the use of the pertaining transformations) $\underline{\tau} = 0$, $\overline{\tau} = 1$ and $\overline{U}_l = 0$, with no loss of generality.

Understandably, as τ_l defines the *maximum time allotted to Black* that player l is willing to accept to play on the White side, we are tempted to parallel this variable to the *valuation* v_l usual in Auction Theory literature, as the maximum price the buyer is willing to pay for an object under auction. So, we may think of the possibility of arranging auction mechanisms to assign sides and time in a reasonable way, selling "the white side" at a given time for Black. However, there is an important caveat to this argument: Unlike traditional auctions, where losing the buy gives always the same result, here τ_l also represents the minimum time to receive that player l is eager to accept to play as Black. In a way, in this auction what is at stake is not a good, but rather the entitlement to obtain one right (to play White) over another (take the black pieces). Hence, losing the auction at a high price (i. e. playing with Black and much time on the clock) is valuable; so, interestingly enough, we will see for instance that the best strategy in an English auction involves overbidding.

Assigning mechanisms

In this scenery, we study a series of mechanisms that assign time and sides. We say that a mechanism $\mathfrak{M}$ is a mapping from a signaling domain $S_1 \times S_2$ to $\mathcal{L}\left([0;1]^3\right)$ (the space of lotteries of triples belonging to $[0;1]^3$), so that for any pair of signals $(s_1, s_2) \in S_1 \times S_2$, $\mathfrak{M}$ assigns a [possibly degenerate] lottery of triples of the form $\left(\delta_1^W, \delta_2^W, t\right)$.

In this work, we will go over some classical mechanisms of assignment, described below:

Fixed Rule (traditional[3]): A fixed time $\bar{t}$ is allotted to Black. A drawing of colors is then performed to assign the right to choose side to one player.

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[1] In an abuse of notation, somewhat disrespectful of order, we will use the sub-index combination $_{l,-l}$ in some ordered pairs.

[2] Cf. note **4**.

[3] For instance, the FIDE Reunification Championship Match between Veselin Topalov and Vladimir Kramnik stated that in case of tie after a 4-game rapid tie-break mini-match, an Armageddon game be played between the players, assigning 5 minutes to the white side and 4 to Black.

Here the signalling domain can be taken to be $S_l = \{0;1\}$. Half the times, the mechanism will assign the triple $(s_1, 1-s_1, \bar{t})$, and half the other times the triple $(1-s_2, s_2, \bar{t})$.

Progressive [English[4]] Auction ***[PA]***: The players stand before a clock running up starting from $\underline{\tau}$. The first player to withdraw from the buy takes Black, and is allotted the time read in the clock.

The space of signals is given by a withdrawal indication at time *t*; however, this is strategically equivalent to take $S_l = [0;1]$, if we interpret s_i as the maximum time at which agent *l* is willing to remain in the auction. We may so interpret $s_l = b_l(\tau_l)$ as the strategy followed by player *l* of type τ_l. The mechanism then assigns the triple

$$(\delta_1^W, \delta_2^W, \hat{t}), \text{ where } \hat{t} = \min(s_1, s_2), \text{ and } \delta_i^W = \begin{cases} 1 & \text{if } s_1 = \max(s_1, s_2) \\ 1/2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}.$$

Dutch Auction ***[DA]***: The players stand before a clock running down starting from $\bar{\tau}$. The first player to stop the clock takes White, the time allotted to Black being the reading of the stopped clock.

The space of signals for each agent is given by $S_l = [0;1]$. $s_l = b_l(\tau_l)$ is the time at which the player *l* of type τ_l is willing to stop the clock. The mechanism assigns the triple

$$(\delta_1^W, \delta_2^W, \hat{t}), \text{ where } \hat{t} = \max(s_1, s_2), \text{ and } \delta_i^W = \begin{cases} 1 & \text{if } s_1 = \max(s_1, s_2) \\ 1/2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}.$$

First-time Sealed Bid ***[FSB]***: The agents submit sealed time offers to stamp on Black's clock. Bids are then uncovered, and the player who assigned the more time to the black side takes white. Time assigned to Black is set to his bid.

The space of signals for each agent is again $S_l = [0;1]$, their strategies being $s_l = b_l(\tau_l)$. The mechanism fixes the triple

$$(\delta_1^W, \delta_2^W, \hat{t}), \text{ where } \hat{t} = \max(s_1, s_2), \text{ and } \delta_i^W = \begin{cases} 1 & \text{if } s_1 = \max(s_1, s_2) \\ 1/2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}.$$

Second-time Sealed Bid ***[SSB]***: This auction is quite similar to the previous one: sealed bids are received. The player assigned the white side is determined exactly as above. However, time allotted to Black is the lower time bid. The outcome of the mechanism is then the triple

$$(\delta_1^W, \delta_2^W, \hat{t}), \text{ where } \hat{t} = \min(s_1, s_2), \text{ and } \delta_i^W = \begin{cases} 1 & \text{if } s_1 = \max(s_1, s_2) \\ 1/2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}.$$

We will now describe two other mechanisms that can be considered in regard to this assignment, the first of the family of sealed-bid auctions, the last quite simple:

[4] This is not the canonical description of an English Auction, of course, though it is perfectly equivalent to the traditional auctions typically portrayed, in which bidders call offers aloud. Since the number of individuals taking part in the auction is only two, withdrawal of a player does not give the remaining agent any significant piece of information, since it merely finishes the auction. Thus, the assumption of private types could be dropped without affecting description and investigation of these mechanisms.

Average-time Sealed Bid *[ASB]*: The agents submit sealed time offers to stamp on Black's clock. Bids are then uncovered, and the player who assigned the more time to the black side, takes white. Yet time assigned to Black is set to the average of both bids.

The mechanism fixes the triple

$$\left(\delta_1^W, \delta_2^W, \hat{t}\right), \text{ where } \hat{t} = \frac{s_1 + s_2}{2}, \text{ and } \delta_i^W = \begin{cases} 1 & \text{if } s_1 = \max(s_1, s_2) \\ 1/2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}.$$

Split-or-Pick[5] *[SrP]*: A draw is held to determine which player calls time, and who calls sides. The player entitled to call time offers how much time for Black to be given. Then, the other player decides which side to take.

The mechanism randomly assigns which player takes which decision: so *time* and *side* are chosen at random from $\{1;2\}$. The first player to send a signal is the one to choose time. The space of signals for the first individual is $S_{time} = [0;1]$. For the other player, we have $S_{side} = \{0;1\}$. The outcome is then the triple

$$\left(\delta_1^W, \delta_2^W, \hat{t}\right), \text{ where } \hat{t} = s_{time},\ \delta_{side}^W = s_{side} \text{ and } \delta_{time}^W = 1 - s_{side}.$$

2 MAIN RESULTS

2.1 Equivalence of mechanisms

Here we comment on a commonplace in Auction Theory: equivalence of auction mechanisms.

Theorem 1: *In the present model, [**DA**] and [**FSB**] are strategically equivalent.*

Proof: Trivial, as signaling spaces and outcome functions coincide.

*Corollary of Theorem 1: Every equilibrium of [**DA**] matches one of [**FSB**] and vice versa.*

Hence, there is no need for analyzing these two auctions independently.

Theorem 2: *In the present model, [**PA**] and [**SSB**] are strategically equivalent.*

Proof: Trivial, parallels verbatim the one for the previous theorem.

We may deduce a corollary similar to the preceding one.

2.2 Obtaining equilibria

Under two main assumptions (conjecture of a symmetric, monotonic equilibrium[6] and restriction of signals to fall into the interval he interval $[0;1]$) the following bidding functions can be deduced from first-order conditions of the agents' optimization problems:

[FSB]:

Under the symmetry assumption, first-order condition translates into the differential equation

$$F(\tau_l) + \frac{2\left(B(\tau_l) - \tau_l\right) f(\tau_l)}{B'(\tau_l)} = 0 \qquad \textbf{[2]}$$

[5] The choice of name for this mechanism results from resemblance to a popular method of dividing a bar (for instance, of chocolate) between two persons: it compels one individual to split the bar and permits the other one to choose a portion.

[6] Under the monotonicity hypothesis, the assumed restriction on the signal space translates into the boundary conditions $B(0) \geq 0$ and $B(1) \leq 1$. Though this restriction may seem too demanding at first sight, notice that the violation of that requirement ordinarily implies the existence of an unbounded equilibrium, which is of not much interest.

We solve for **[2]**, to obtain

$$B(\tau_l)=\frac{K+\int_{\tilde{\tau}}^{\tau_l}2y\cdot F(y)f(y)\mathrm{d}y}{F^2(\tau_l)} \qquad \textbf{[3]}$$

From the fact that signal $B(\tau_l)$ is deemed to fall inside $S_l=[0;1]$, it is inferred that $K=0$. Monotonicity, in turn, implies that $\tilde{t}=0$. Thus we have

$$B(\tau_l)=\frac{\int_0^{\tau_l}2y\cdot F(y)f(y)\mathrm{d}y}{F^2(\tau_l)}=\tau_l-\frac{\int_0^{\tau_l}F^2(y)\mathrm{d}y}{F^2(\tau_l)} \qquad \textbf{[4]}$$

…where the last expression results from integrating by parts the middle term. If the uniform distribution is assumed, **[4]** simplifies into

$$B(\tau_l)=\frac{2}{3}\tau_l \qquad \textbf{[5]}$$

Repeating this procedure for the other mechanisms, the following bidding functions are found:

[SSB]:

$$B(\tau_l)=\tau_l+\frac{\int_{\tau_l}^{1}(1-F(y))^2\,\mathrm{d}y}{(1-F(\tau_l))^2}=\tau_l-\frac{\int_1^{\tau_l}(1-F(y))^2\,\mathrm{d}y}{(1-F(\tau_l))^2} \qquad \textbf{[6]}$$

For the case of the uniform distribution, this translates into

$$B(\tau_l)=\frac{1}{3}+\frac{2}{3}\tau_l \qquad \textbf{[7]}$$

[ASB]:

$$B(\tau_l)=\frac{K-\int_0^{\tau_l}2y\left(1/2-F(y)\right)f(y)\mathrm{d}y}{\left(1/2-F(y)\right)^2} \qquad \textbf{[8]}$$

A value for integration constant K must be found so that monotonicity and boundary conditions are fulfilled. Letting $\tilde{\tau}$ stand for the median of the distribution [i. e. $\tilde{\tau}=F^{-1}\left(1/2\right)$], we can rewrite **[8]** as

$$B(\tau_l)=\frac{K-\int_0^{\tilde{\tau}}2y\left(1/2-F(y)\right)f(y)\mathrm{d}y}{\left(1/2-F(\tau_l)\right)^2}-\frac{\int_{\tilde{\tau}}^{\tau_l}2y\left(1/2-F(y)\right)f(y)\mathrm{d}y}{\left(1/2-F(\tau_l)\right)^2} \qquad \textbf{[9]}$$

Since $B(\tau_l)$ is assumed to belong to the unit interval[7], it must be the case that $K=\int_0^{\tilde{\tau}}2y\left(1/2-F(y)\right)f(y)\mathrm{d}y$, and thus, **[8]** finally turns into

$$B(\tau_l)=\frac{-\int_{\tilde{\tau}}^{\tau_l}2y\left(1/2-F(y)\right)f(y)\mathrm{d}y}{\left(1/2-F(\tau_l)\right)^2}=\tau_l-\frac{\int_{\tilde{\tau}}^{\tau_l}(1-2F(y))^2\,\mathrm{d}y}{(1-2F(\tau_l))^2} \qquad \textbf{[10]}$$

For the uniform case, $\tilde{\tau}=1/2$, and **[10]** translates into

$$B(\tau_l)=\frac{1}{6}+\frac{2}{3}\tau_l \qquad \textbf{[11]}$$

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[7] Or simply, presumed bounded; see note 6.

[SrP]:

For a fixed signal s_{time} chosen by agent *time*, individual *side* has a straightforward reaction: she takes white if and only if $s_{time} \le \tau_{side}$.

Agent *time*'s expected utility is

$$u_{time}(\tau_{time}, b) = \overline{U}_{time} + F(b)(\tau_{time} - b) + (1 - F(b))(b - \tau_{time}) \qquad \textbf{[12]}$$

First-order condition implies that

$$F(b) + (b - \tau_{time}) f(b) = \tfrac{1}{2} \qquad \textbf{[13]}$$

Equation **[13]** cannot be explicitly solved in terms of a general [cumulative] probability function $F(\cdot)$. However, it is straightforward that for the median $\widetilde{\tau} = F^{-1}(\tfrac{1}{2})$, $B(\widetilde{\tau}) = \widetilde{\tau}$; and for any other type τ_{time}, $B(\tau_{time}) - \tau_{time}$ must have sign opposite to that of $\tau_{time} - \widetilde{\tau}$.

Let us take the uniform case. We solve **[13]** for this particular instance, to find equilibrium strategy for agent *time*:

$$\boxed{B(\tau_l) = \frac{1}{4} + \frac{1}{2}\tau_l} \qquad \textbf{[14]}$$

3 MECHANISM DESIGN ASPECTS

3.1 Basics

Let us restate a mechanism as a rule $m(s_1(\tau_1), s_2(\tau_2))$ that sets for each pair of signals $s_1(\tau_1), s_2(\tau_2)$ an outcome $(\delta_1^W, \delta_2^W, \hat{t})$. We define $\delta_1^W(s_1(\tau_1), s_2(\tau_2))$, $\delta_2^W(s_1(\tau_1), s_2(\tau_2))$ and $\hat{t}(s_1(\tau_1), s_2(\tau_2))$ as the respective projections of $m(\cdot)$. We may focus our attention on a particular *BNE* of the mechanism, say $(s_1(\tau_1), s_2(\tau_2))$.

A *direct mechanism* is one for which $S_1 \times S_2 = [0;1]^2$ (the space of signals is precisely the space of types). A Bayesian *direct revelation mechanism* is a direct mechanism with a Bayesian Nash equilibrium [*BNE*] in which agents fix $s_l = \tau_l$ (strategy at equilibrium is signaling own type). The Revelation Principle[8] states that for any Bayesian Nash equilibrium $\widetilde{m}$ of a mechanism $\mathfrak{M}$, there exists a direct revelation mechanism $\mathfrak{M}^d$ that assigns in a pertaining *BNE* $\widetilde{m}^d$ an outcome matching that of $\widetilde{m}$. Note that most of the mechanisms analyzed above are direct, though they are not direct revelation mechanisms, for equilibrium strategies diverge from type truth-telling. However, it is easy to verify that any of them could be associated to a direct revelation mechanism, through a suitable mapping from signals to types.

We can, for [each equilibrium of] the respective mechanism, define the *ex post* utility for agent *l*:

$$\hat{u}_l(\tau_l, \tau_{-l}) = \delta_l^W(\tau_l, \tau_{-l}) \cdot (\tau_l - t(\tau_l, \tau_{-l})) + \delta_{-l}^W(\tau_l, \tau_{-l}) \cdot (t(\tau_l, \tau_{-l}) - \tau_l) \qquad \textbf{[15]}$$

…where $\delta_l^W(\cdot)$ and $t(\cdot)$ represent the pair of outcomes determined by the mechanism. The *interim* utility is

$$\widetilde{u}_l(\tau_l) = Q_l(\tau_l) \cdot (\tau_l - T_l^W(\tau_l)) + (1 - Q_l(\tau_l)) \cdot (T_l^B(\tau_l) - \tau_l) \qquad \textbf{[16]}$$

…where $Q_l(\tau_l) = \int_0^1 \delta_l^W(\tau_l, \tau_{-l}) f(\tau_{-l}) \mathrm{d}\tau_{-l}$ is the expected probability of playing with the

[8] See, for instance, Mas-Colell, Whinston and Green (1995). As stated, this is a revelation principle to be applied to Bayesian Nash equilibria. A similar construction deals with dominant-strategy equilibria.

white pieces (i.e. winning the auction), and $T_l^W(\tau_l)$ and $T_l^B(\tau_l)$ represent the expected time allotted when having the white and black pieces, respectively:

$$T_l^W(\tau_l) = \frac{\int_0^1 t(\tau_l, \tau_{-l})\delta_l^W(\tau_l, \tau_{-l}) f(\tau_{-l})\mathrm{d}\tau_{-l}}{\int_0^1 \delta_l^W(\tau_l, \tau_{-l}) f(\tau_{-l})\mathrm{d}\tau_{-l}}$$

and

$$T_l^B(\tau_l) = \frac{\int_0^1 t(\tau_l, \tau_{-l})\left(1 - \delta_l^W(\tau_l, \tau_{-l})\right) f(\tau_{-l})\mathrm{d}\tau_{-l}}{\int_0^1 \left(1 - \delta_l^W(\tau_l, \tau_{-l})\right) f(\tau_{-l})\mathrm{d}\tau_{-l}}$$

Finally, *ex ante* utility is given by

$$\breve{u}_l = \int_0^1 \tilde{u}_l(\tau_l) f(\tau_l)\mathrm{d}\tau_l$$

$$= \int_0^1 \left[Q_l(\tau_l)\cdot\left(\tau_l - T_l^W(\tau_l)\right) + \left(1 - Q_l(\tau_l)\right)\cdot\left(T_l^B(\tau_l) - \tau_l\right)\right] f(\tau_l)\mathrm{d}\tau_l \qquad \textbf{[17]}$$

The usual definition of incentive compatibility applies to our model: a direct mechanism satisfies incentive compatibility[9] ***[IC]*** if it is a direct revelation mechanism, that is, no agent can do better signaling anything other than her true type, while the other agent carries out that honest strategy.

3.2 Implementation

A *social function* $\Phi(\tau_1, \tau_2)$ is a mapping that assigns for each pair of [revealed or known] types of players an outcome. It thus represents a series of normative principles applied to the assignment problem under study. We may build a social function satisfying a number of prescriptive properties and then ask whether the assignment laid down by this social mapping could be achieved [under incomplete information] through a certain mechanism. We express that a mechanism *implements*[10] some social function $\Phi(\tau_1, \tau_2)$ [through a certain *BNE*] if for any pair (τ_1, τ_2) the outcomes of the mechanism and the function $\Phi(\cdot)$ coincide.

A similar though simpler and broader method is to imprint some normative principle onto a property to be fulfilled by a specific mechanism (under a certain *BNE*).

- Equality

It is reasonable to demand that any mechanism treat equally both players. But this is such a vague statement: what is to treat both players equally? A first proposition to delimit this impression is the following:

  - Anonymity ***[ANON]***:
    $\forall s_1 \forall s_2,\ m(s_1(\cdot), s_2(\cdot)) = \left(\hat{\delta}_1^W, \hat{\delta}_2^W, \hat{t}\right) \Rightarrow m(s_2(\cdot), s_1(\cdot)) = \left(\hat{\delta}_2^W, \hat{\delta}_1^W, \hat{t}\right)$

(For any pair of signals, which player emits which signal is irrelevant to the outcome of the mechanism.)

It is straightforward that all the analyzed mechanisms satisfy anonymity.

A much stronger characterization of equal treatment is depicted in the next property:

  - Mid-time assignment ***[mTIME]***:
    $\forall \tau_1 \forall \tau_2,\ \hat{t}(\cdot) \in [\min(\tau_1, \tau_2); \max(\tau_1, \tau_2)]$

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[9] Again, this is a characterization for *BNE* (i. e., weaker than incentive compatibility in dominant strategies).

[10] Once more, this is *Bayesian* implementation.

This (quite strong)[11] property determines that the time for Black be set into the bargaining interval $[\min(\tau_1,\tau_2);\max(\tau_1,\tau_2)]$, being no agent neither excessively benefited nor harmed by the assignment. Notice that none of the examined mechanisms fulfils this property.

- Efficiency

Let us consider two generic direct mechanisms, $\mathfrak{M}^d$ and $\mathfrak{N}^d$. We define $\breve{u}_l^{\mathfrak{L}}$, $\tilde{u}_l^{\mathfrak{L}}(\tau_l)$ and $\hat{u}_l^{\mathfrak{L}}(\tau_l,\tau_{-l})$ as the *ex ante*, *interim* and *ex post* [respectively] utility under the mechanism $\mathfrak{L}$. Let us take a (Lebesgue) measure μ over the power set of the unit interval[12]. We say that $\mathfrak{M}^d$ *ex ante* Pareto-dominates $\mathfrak{N}^d$ if $\forall l=1,2$ $\breve{u}_l^{\mathfrak{M}^d}\geq\breve{u}_l^{\mathfrak{N}^d}$ and $\exists l \mid \breve{u}_l^{\mathfrak{M}^d}>\breve{u}_l^{\mathfrak{N}^d}$. Additionally, we state that $\mathfrak{M}^d$ *interim* Pareto-dominates $\mathfrak{N}^d$ if $\forall l=1,2$, $\tilde{u}_l^{\mathfrak{M}^d}(\tau_l)\geq\tilde{u}_l^{\mathfrak{N}^d}(\tau_l)$ almost everywhere, and $\exists l,\exists \mathrm{T}_l=\left\{\tau_l \mid \tilde{u}_l^{\mathfrak{M}^d}(\tau_l)>\tilde{u}_l^{\mathfrak{N}^d}(\tau_l)\right\} \mid \mu(\mathrm{T}_l)>0$ [the set T of types strictly better off under $\mathfrak{M}^d$ does not have null measure]. Finally, $\mathfrak{M}^d$ *ex post* Pareto-dominates $\mathfrak{N}^d$ if for $l=1,2$, $\hat{u}_l^{\mathfrak{M}^d}(\tau_l,\tau_{-l})\geq\hat{u}_l^{\mathfrak{N}^d}(\tau_l,\tau_{-l})$ almost everywhere in $[0;1]^2$, and $\exists l,\exists \mathrm{T}_l=\left\{\tau_l \mid \exists\tau_{-l},\hat{u}_l^{\mathfrak{M}^d}(\tau_l,\tau_{-l})>\hat{u}_l^{\mathfrak{N}^d}(\tau_l,\tau_{-l})\right\} \mid \mu(\mathrm{T}_l)>0$. As usual in the literature, we define a series of efficiency concepts regarding the previous definitions: A direct mechanism $\mathfrak{M}^d$ is said to be *ex ante/interim/ex post* efficient if there exists no mechanism $\mathfrak{N}^d$ that *ex ante/interim/ex post* (respectively) Pareto-dominates $\mathfrak{M}^d$. In other words, a mechanism is *ex ante* efficient if it assures the maximum average utility for each generic agent. *Interim* efficiency typifies a situation in which no agent of any type can be made better off without harming welfare of the same agent when being of another type (or the other player). Lastly, *ex post* efficiency rules out the possibility of agents of respective types τ_1,τ_2 to simultaneously improve their welfare through bargaining from the outcome of the mechanism. It is quite straightforward[13] that *ex ante* efficiency implies *interim* efficiency; and from the compliance of the latter, *ex post* efficiency could be deduced.

These definitions on efficiency could be narrowed to consider only some type of mechanisms[14]: $\mathfrak{M}^d\in\mathscr{M}$ is said to be *ex ante/interim/ex post* efficient (over a family of direct mechanisms $\mathscr{M}$) if there exists no mechanism $\mathfrak{N}^d\in\mathscr{M}$ that *ex ante/interim/ex post* (respectively) Pareto-dominates $\mathfrak{M}^d$. It is frequent to speak of incentive-efficient mechanisms when analysis is restricted only to the family $\mathscr{M}_{IC}$ of direct revelation mechanisms. We will somewhere below refer to the family of *feasible* mechanisms $\mathscr{M}_F$.

The following two lemmas further characterize some aspects of efficiency:

Lemma 1: If a direct mechanism $\mathfrak{M}^d$ maximizes aggregate *ex ante* utility $\sum_{i=1}^{2}\breve{u}_i$ over a family of mechanisms $\mathscr{M}$, then it is [*ex ante*] efficient over $\mathscr{M}$.

Proof: Trivial.

[11] We note that welfare of the players is not only influenced by time assigned, but also by the (possibly contingent) side allotted. So, for instance, for the case of $\tau_1=0.2$, $\tau_2=0.4$, an assignment $(\delta_1^W,\delta_1^W,\hat{t})=(0,1,0.3)$ gives the same expected utility for each player that $(\delta_1^W,\delta_1^W,\hat{t})=(1/3,2/3,0.5)$ does.

[12] This is in order to dispose of comparisons over type sets of null measure, for the sake of clarity.

[13] It is a matter of integrating premises on a *reductio ad adsurdum*.

[14] An important caveat is that the aforementioned chain of efficiency implications [cf note **13**] apply to a certain family of mechanisms, but do not extend over a more general set: an *interim* incentive efficient mechanism need not be *ex post* (globally) efficient.

Lemma 2: Consider the family[15] $\mathscr{F}$ of direct mechanisms with outcome functions $\left(\delta_1^W(\tau_1,\tau_2),\delta_2^W(\tau_1,\tau_2),\hat{t}(\tau_1,\tau_2)\right)$ such that for any pair of agents' types τ_1,τ_2, we have that $\delta_l^W(\tau_1,\tau_2)+\delta_{-l}^W(\tau_1,\tau_2)=1$. Then the two following propositions are equivalent:

1. For the mechanism $\mathfrak{M}^d \in \mathscr{F}$, $\tau_1 \neq \tau_2$ implies almost everywhere that for $l=1,2$ $\delta_l^W(\tau_1,\tau_2)\in\{0;1\}$, and $\delta_l^W(\tau_1,\tau_2)=1 \Leftrightarrow \tau_l=\max(\tau_1,\tau_2)$.
2. The mechanism $\mathfrak{M}^d$ is *ex post* efficient over $\mathscr{F}$.

In words, a mechanism (on $\mathscr{F}$!) is *ex post* efficient if and only if it [almost always] allots the white pieces to the player of higher type.

Proof: See Mathematical Appendix.

Under the studied monotonic *BNE*, both ***[FSB]*** and ***[SSB]***, as well as ***[ASB]***, are *ex post* efficient (over $\mathscr{F}$). They always assign White to the higher-typed player, whatever time for Black is set. However, ***[SrP]*** cannot be *ex post* efficient over that family, since for any type τ_{time} other than the median $\tilde{\tau}$, an interval exists of the form $(B(\tau_{time});\tilde{\tau})$ or $(\tilde{\tau};B(\tau_{time}))$ for which for any *side* type τ_{side} falling inside the interval, an *ex post* inefficient assignation arises.

3.3 *Interim* characterization of mechanisms

It is straightforward that two distinct mechanisms may have one concurrent direct mechanism that maps their equilibria. Furthermore, it is also clear that two distinct direct mechanisms which assign different *ex post* outcomes, could yield the same *interim* utility for the players. Since we are dealing with *BNE* as the main equilibrium concept, we may restrict our attention to *interim* outcomes of mechanisms.

Hence, in the first place we will focus on the *interim* outcome of a direct mechanism given by the 6-uple $\left(Q_1(\tau_1),T_1^W(\tau_1),T_1^B(\tau_1),Q_2(\tau_2),T_2^W(\tau_2),T_2^B(\tau_2)\right)$. In fact, we notice that **[16]** can be rewritten as

$$\tilde{u}_l(\tau_l)=\tilde{Q}_l(\tau_l)\cdot\tau_l-\tilde{T}_l(\tau_l) \qquad \textbf{[18]}$$

…where $\tilde{Q}_l(\tau_l)=2Q_l(\tau_l)-1$ and $\tilde{T}_l(\tau_l)=Q_l(\tau_l)\cdot T_l^W(\tau_l)+(Q_l(\tau_l)-1)\cdot T_l^B(\tau_l)$ are the ***Side*** and ***Time*** *interim* functions. Expression **[18]** is a formulation quite similar to the linear *interim* expression for canonical auctions.

Thus, the *interim* properties of a mechanism rest merely on the quadruple $\left(\tilde{Q}_1(\tau_1),\tilde{T}_1(\tau_1),\tilde{Q}_2(\tau_2),\tilde{T}_2(\tau_2)\right)$. Nevertheless, we are interested only in relevant mechanisms that fulfill a series of constraints. We say that a direct mechanism $\mathfrak{M}^d$ is [strongly] ***feasible*** if it satisfies the following three properties:

[IC] (Incentive compatibility, as stated above): $\forall\tau_l\ \tilde{u}_l(\tau_l)=\operatorname*{Sup}_{t\in[0;1]}\hat{u}_l(\tau_l,t)$, where $\hat{u}_l(\tau_l,t)=\tilde{Q}_l(t)\cdot\tau_l-\tilde{T}_l(t)$.

[qDEC] (feasible *side* decomposition):

$$\exists q_1(\tau_1,\tau_2),q_2(\tau_1,\tau_2)\mid \tilde{Q}_l(\tau_l)=\int_0^1 q_l(\tau_1,\tau_2)f(\tau_{-l})\mathrm{d}\tau_{-l}$$
$$\wedge\forall(\tau_1,\tau_2)\in[0;1]^2\ \left(0\le q_l(\tau_1,\tau_2)\le 1\wedge q_l(\tau_1,\tau_2)=1-q_{-l}(\tau_1,\tau_2)\right)$$

[tDEC] (feasible *time* decomposition):

[15] $\mathscr{F}$ is the family of mechanisms that set feasible *time* and *side* assignments; notice it does not contain only direct revelation mechanisms.

$$\exists t_1(\tau_1,\tau_2),t_2(\tau_1,\tau_2) \mid \widetilde{T}_l(\tau_l)=Q_l(\tau_l)T_l^W(\tau_l)+(Q_l(\tau_l)-1)T_l^B(\tau_l)$$
$$\wedge \forall l=1,2\ \forall(\tau_1,\tau_2)\in[0;1]^2\ (0\le t_l(\tau_1,\tau_2)\le 1 \wedge t_l(\tau_1,\tau_2)=t_{-l}(\tau_1,\tau_2))$$

…where $T_l^W(\tau_l)=\dfrac{\underset{\tau_{-l}}{\mathrm{E}}\left[t_l(\tau_l,\tau_{-l})\cdot q_l(\tau_l,\tau_{-l})\right]}{Q_l(\tau_l)}$, the *Time* function defined above.

A useful re-expression of *[IC]* is given in the next theorem:

***<u>Theorem 3</u>**: Consider a direct mechanism $\mathfrak{M}^d$ for which the function $\widetilde{Q}_l(\cdot)$ as described above is quasicontinuous (i. e. is a bounded, piecewise continuous function). Then, the following three propositions are equivalent:*

a) $\mathfrak{M}^d$ satisfies [**IC**].

b) $\widetilde{u}_l(\cdot)$ under $\mathfrak{M}^d$ is convex, and $\widetilde{u}_l(\tau_l)\equiv\int_0^{\tau_l}\widetilde{Q}_l(y)\mathrm{d}y-\widetilde{T}_l(0)$.

c) $\widetilde{Q}_l(\tau_1)\ge\widetilde{Q}_l(\tau_2)\Leftrightarrow\tau_1\ge\tau_2$, and $\widetilde{u}_l(\tau_l)\equiv\int_0^{\tau_l}\widetilde{Q}_l(y)\mathrm{d}y-\widetilde{T}_l(0)$.

Proof: See Mathematical Appendix.

As a matter of fact, it can be shown that *[IC]* implies the quasicontinuity of $\widetilde{Q}_l(\cdot)$: It has already been established that the fulfillment of *[IC]* implies the convexity of $\widetilde{u}_l(\cdot)$, and a convex function can only have a finite number of kinks in $[0;1]$.

We will now turn our attention to some relations between utility, *Side* and *Time* functions. The next theorem contains 4 propositions: The first two clauses refer to a very well known association between *Side* and utility functions. Although the *Time* function (at least, evaluated at the minimum type value) seems to play a role in the determination of *ex ante* utility, this is no longer the case under anonymity: the last two propositions address the issues of what constrains this condition imposes on $\widetilde{T}_l(\cdot)$.

***<u>Theorem 4</u>**: For any two feasible mechanisms $\mathfrak{M}^I$ and $\mathfrak{M}^{II}$, the following propositions are true (employing superindices for the pertaining functions under those rules):*

1. *For any* l,
   $\widetilde{Q}_l^I(\tau_l)\equiv\widetilde{Q}_l^{II}(\tau_l)$ *almost everywhere* $\Leftrightarrow$ $\widetilde{u}_l^{II}(\tau_l)\equiv\widetilde{u}_l^I(\tau_l)+\widetilde{T}_l^I(0)-\widetilde{T}_l^{II}(0)$.
2. *For any* l,
   $\widetilde{Q}_l^I(\tau_l)\equiv\widetilde{Q}_l^{II}(\tau_l)$ *almost everywhere* $\Rightarrow\breve{u}_l^{II}=\widetilde{u}_l^I(\tau_l)+\widetilde{T}_l^I(0)-\widetilde{T}_l^{II}(0)$.
3. *If* $\mathfrak{M}^I$ *satisfies [**ANON**], then* $\breve{u}_1^I=\breve{u}_2^I=\breve{u}^I=\int_0^1\tau\cdot\widetilde{Q}^I(\tau)f^I(\tau)\mathrm{d}\tau$*, and* $\widetilde{T}^I(\cdot)$ *is restrained by the condition* $\int_0^1\widetilde{T}^I(\tau)f^I(\tau)\mathrm{d}\tau=0$.
4. *Under a mechanism* $\mathfrak{M}^I$ *satisfying [**ANON**],*
   $\widetilde{T}^I(0)=\int_0^1\left\{\int_0^\tau\widetilde{Q}^I(y)\mathrm{d}y-\tau\cdot\widetilde{Q}^I(\tau)\right\}f^I(\tau)\mathrm{d}\tau=\int_0^1\int_0^\tau y\cdot\widetilde{Q}'^I(y)f^I(\tau)\mathrm{d}y\,\mathrm{d}\tau$

The proof of this theorem is to be found in the mathematical appendix.

<u>Corollary to Theorem 4</u>: *In the present model, any of the studied auctions [through the respective monotonic equilibria] generate equivalent* ex ante *and* interim *utilities.*

Proof: Since all these auctions satisfy anonymity, their *ex ante* utility must coincide, because their *Side* functions are $\widetilde{Q}(\tau)=F(\tau)$ [see first part of the third proposition from the preceding theorem]. On the other hand, their *interim* utility must only differ in a constant. However, that constant is zero, for the *Time* function evaluated at the minimum type (as shown in the last proposition of the theorem) coincides for all auctions.

3.4 Efficiency revisited

Two vexed questions arise: In the first place, we may ask whether there is a way of finding an *ex ante* efficient mechanism with a fairer behavior[16] than the analyzed auctions; and finally, if the answer to this question is negative[17], whether there exists at all a rule that (at least *ex ante*) Pareto-dominates the family of auction mechanisms. The next theorem replies these two inquiries:

Theorem 5: *In the present model, the family of auction rules belongs to the set of* ex ante *feasibly efficient mechanisms. Furthermore, auctions are not only* ex ante *efficient over the set of feasible mechanisms: they cannot be Pareto-dominated by any other rule contained in* $\mathscr{F}$ *(the set described in note **15**)*[18].

Again we refer the reader to the Mathematical Appendix for a proof of the theorem.

We notice that none of the studied auctions are direct revelation mechanisms: equilibrium strategies involve some computation, and signaling entails only an indirect representation of type. It is natural to ask if a practically implementable, non Pareto-inferior mechanism can be devised. For the uniform case, we present a peculiar rule that *partially* fulfils all these specifications:

<u>Example 2</u>: With $f(\tau)=1$ on support $[0;1]$, consider the mechanism $\mathscr{B}$ defined by the following algorithm:

i) $i=1$, $a_1=0$ and $b_1=1$ are set as initial values.

ii) $\hat{t}_i = k2^{-i}$ is set, where $k \in Z$ is chosen so that $k2^{-i} \in (a_i;b_i)$.

iii) Players call option of side: "White", "Black" or "indifferent".

iv) If the two signals are "White", the rule sets $a_{i+1}=\hat{t}_i$, $b_{i+1}=b_i$. The algorithm iterates on *i*, and proceeds again to step ii).

v) If the two signals are "Black", the mechanism fixes $a_{i+1}=a_i$, $b_{i+1}=\hat{t}_i$. The algorithm iterates on *i*, receding to step ii).

vi) If none of the preceding two cases arise, $\hat{t}$ is finally set to $\hat{t}_i = k2^{-i}$, and the assignment of colours follow the signals given by the players.

Additionally, consider the mechanism $\mathscr{B}'$ given by the next procedure:

The players privately[19] signal times τ_1 and τ_2.

The mechanism assigns time in the following manner:

First i is calculated as $\min\left(j \in Z \mid \exists k \in Z, k2^{-j} \in \left[\underline{\tau};\overline{\tau}\right]\right)$, with $\underline{\tau}=\min(\tau_1,\tau_2)$ and $\overline{\tau}=\max(\tau_1,\tau_2)$.

Then time is set as $\hat{t}=k2^{-i}$, with $k \in Z \mid k2^{-j} \in \left[\underline{\tau};\overline{\tau}\right]$, if there exists only one such *k*; or

[16] *Fairer* means in this context a mechanism that performs better in terms of egalitarian type evaluation, for instance outperforming another rule as measured by a social reference function like $\int_0^1 \tilde{u}_l^{\alpha}(\tau_l)\mathrm{d}\tau_l$, for $0<\alpha<1$.

[17] Thus, allowing for mechanisms that fail to meet *[**ANON**]*, for instance.

[18] It is not surprising then that the mechanism $\mathscr{O}^d$ described in example 1 , which even violates *[**IC**]*, cannot perform better in *ex ante* terms than any auction.

[19] Through a sealed bid, for instance.

$\hat{t} = k2^{-i}$, with $k \in Z \mid k2^{-j} \in (\underline{\tau};\overline{\tau})$, otherwise.

Finally, if $\tau_1 \neq \tau_2$ sides are assigned so that $\delta_l^W = 1 - \delta_{-l}^W$ and $\delta_l^W = 1 \Leftrightarrow \tau_l = \overline{\tau}$; if $\tau_1 = \tau_2$, sides are set at random.

Though it is not explicitly proven in this paper, equilibrium strategies for $\mathcal{B}$ can be in the end simplified to functions $b_l(\tau_l)$ that set a reference time: for those times over this mark, the player will call "White" and "Black" will be announced whenever the mark falls over the specified time. It could also be proven that $b_l(\tau_l) = \tau_l$ almost everywhere, which is a consequence of the strategic equivalence[20] of mechanisms $\mathcal{B}$ and $\mathcal{B}'$, and the fact that the latter fulfils *[IC]* almost everywhere, as proven below.

We will focus now on mechanism $\mathcal{B}'$. First we point out a problematic subset of the unit interval, which fortunately has null measure:

<u>*Definition 1*</u>*:* Let us define $\overset{/}{2}$ as the subset of $[0;1]$ of numbers that have a finite binary expansion (the set of multiples of some submultiple of two: e. g. $1/2$, $3/8$, $15/16$, 1, etc. Formally:

$$\overset{/}{2} = \left\{ t \in [0;1] \mid \exists k \in N, \exists \alpha_i \in \{0;1\}, i = 1 \ldots k \mid t = \sum_{i=1}^{k} \alpha_i 2^{-i} \right\} \qquad \textbf{[19]}$$

Notice that the mechanism satisfies *[**ANON**]*. We may thus skip the subindices that indicate players. Now the following theorem addresses the issue of incentive compatibility.

<u>***Theorem 6***</u>: *Over the support* $[0;1]$*,* $\mathcal{B}'$ *fails to fulfill [**IC**].*

Proof: Take a point $\tau^0 \in \overset{/}{2}$. Though the *Side* function $\tilde{Q}(\tau) = \tau$ is continuous everywhere, the functions $T^W(\tau)$ and $T^B(\tau)$ are only continuous to the right and to the left (respectively) evaluated at τ^0. Thus, function $\tilde{u}(\tau)$ is not continuous in $\overset{/}{2}$. But a mechanism satisfying [**IC**] must generate a convex (and thus, continuous) function $\tilde{u}(\tau)$, according to Theorem 3. Thus, $\mathcal{B}'$ cannot fulfill that condition.

It is apparent that the set $\overset{/}{2}$ (of null measure) is clearly problematic when studying the properties of this mechanism. One may wonder which are the effects of restricting the support to the complement of this set on the unit interval:

<u>*Definition 2*</u>*:* Let us define the set $\mathcal{S} = [0;1] \setminus \overset{/}{2}$.

<u>***Theorem 7***</u>: *Restricted to the type domain* $\mathcal{S}$*,* $\mathcal{B}'$ *is a feasible mechanism. Indeed, it also satisfies condition [**mTIME**]. Furthermore,* $\mathcal{B}'$ *is* ex ante *efficient: it yields the same* interim *(and naturally,* ex ante*) utility that any of the auctions.*

Proof: Please see Mathematical Appendix.

4 THE MODEL IN AN ECONOMIC PERSPECTIVE

4.1 A new scheme

Let us consider the next economic example:

[20] The proof of this equivalence is not expounded here, but it is quite straightforward indeed, once the strategy for each player under mechanism $\mathcal{B}$ is established to be a function $b_l(\tau_l)$ as stated. We also note that for certain improbable cases [namely, $b_1(\tau_1) = b_2(\tau_2)$], the mechanism $\mathcal{B}$ never stops iterating. However this mechanism *almost surely* will produce a result in a finite number of steps.

There are two goods, *A* and *B*, which must be allotted to two agents, 1 and 2. We will further suppose there is some means of transfer (basically, money) so as to define payments or exchange flows from one agent to the other. Furthermore, a linear form of utility will be assumed. So for instance if a payment x_l^A from agent *l* (again $l = 1,2$) to the other individual is prescribed when *l* is assigned the good *A*, her utility will be:

$$U_l^A = \pi_l^A - x_l^A \quad \textbf{[20]}$$

…where π_l^A is the net utility or benefit agent *l* gets for receiving the good *A*.

Similarly, for the case of receiving *B* under a payment x_l^B, we have

$$U_l^B = \pi_l^B - x_l^B \quad \textbf{[21]}$$

… π_l^B being defined analogously to π_l^A.

We may in general suppose that a transfer is made from the designed owner of one good to the other individual. Let us define

$t_l^A = x_l^A$ and $t_l^B = -x_l^B$ for $l = 1,2$

Thus we redefine **[20]** and **[21]**:

$$U_l^A = \pi_l^A - t_l^A \quad \textbf{[22]}$$

$$U_l^B = \pi_l^B + t_l^B \quad \textbf{[23]}$$

So, a positive value on those variables means a transfer from the owner of *A* to the agent to whom *B* has been allotted. (These definitions are only for instructional purposes; in principle, no restriction on signs of these variables will be imposed.) We may however speak of rules assigning the good *A* over *B* under this arrangement, whether the assignment implies a payment or compensation[21]. We define *A* to be the *prize* of the allotment, whereas *B* is termed the *residual*.

Let us define τ_l as the transfer that makes individual *l* indifferent between receiving one good or the other: For $t_l^A = -t_l^B$ then, τ_l is the value of t_l^A for which $U_l^A = U_l^B$. Simple algebra yields $\tau_l = \frac{1}{2}\left(\pi_l^A - \pi_l^B\right)$. Thus, taking $\overline{U}_l = \frac{1}{2}\left(\pi_l^A + \pi_l^B\right)$ we get

$$U_l^A = \overline{U}_l + \tau_l - t_l^A \quad \textbf{[24]}$$

$$U_l^B = \overline{U}_l + t_l^B - \tau_l \quad \textbf{[25]}$$

In a parallel fashion to the incomplete information framework described for the chess example (under symmetry and statistical independence), we suppose a type support $\left[\underline{\tau};\overline{\tau}\right]$ over which a [non-degenerate] differentiable cumulative probability distribution $F(\cdot)$ is defined. Valuation independence (i. e. *private types*) is again assumed.

4.2 Results

In general terms, all of the preceding results derived for our chess model apply here as well. One slight difference (according to how the new model has been defined) is simply that transfers have been in principle not meant to coincide, so for given signals $(s_1, s_2) \in S_1 \times S_2$ a mechanism is a quadruple $\left(\delta_1^A, \delta_2^A, t_1, t_2\right)$ rather than a triple. However, the original definition could be seen as a quadruple satisfying the restriction $t_1 = t_2$, which is by the way a completely natural constrain on the mechanisms appropriate for our chess case. However, a

[21] As a matter of fact, the purpose for this arrangement is to make this new frame correspond to the initial setting for our motivational chess example.

less restrictive approach may be in place here since, for instance, a rule specifying a payment by an individual greater in amount than the compensation received by the remaining agent in principle should not be discarded. Hence, to characterize feasibility in this general framework ***[tDEC]*** could be replaced by the condition

[trDEC] (decomposition into feasible[22] *transfer* functions):

$$\exists t_1(\tau_1,\tau_2), t_2(\tau_1,\tau_2) \mid \tilde{T}_l(\tau_l) = Q_l(\tau_l) T_l^W(\tau_l) + (Q_l(\tau_l) - 1) T_l^B(\tau_l)$$

$$\wedge \forall (\tau_1,\tau_2) \in [0;1]^2 \left(t_l(\tau_1,\tau_2) > t_{-l}(\tau_1,\tau_2) \Leftrightarrow q_l(\tau_1,\tau_2) > q_{-l}(\tau_1,\tau_2) \right)$$

For cases in which the functions $q_l(\cdot)$ take fractional values, ***[trDEC]*** simply provides that the expected payment stipulated by the rule exceed the expected compensation to receive.

We notice that ***[qDEC]*** fully applies (after some re-labeling) to our extended scheme: no relaxation on this formulation could be made for a rule to be feasible, since it has been assumed that exactly one of both goods must be allotted to each individual.

Let us define *weak feasibility* as the fulfillment of ***[IC]***, ***[qDEC]***, and ***[trDEC]***. We label the family of *weakly feasible* mechanisms as $\mathscr{W}$. The next lemma aims at showing the robustness of preceding results on efficiency:

Lemma 3: *If a mechanism is* ex ante *feasibly efficient, then it is* ex ante *efficient over* $\mathscr{W}$.

Proof: As stated in the proof of Theorem 5, any *ex ante* feasibly efficient rule must maximize aggregate *ex ante* utility. But it is evident that any mechanism maximizing aggregate *ex ante* utility under ***[IC]***, ***[tDEC]*** and ***[qDEC]*** must do the same under ***[IC]***, ***[qDEC]***, and ***[trDEC]***. Hence, any feasibly efficient rule maximizes ex ante aggregate utility over $\mathscr{W}$, which in turn implies by Lemma 1 that it is ex ante feasible over that family.

QED[23]

We could also think of *fixed rules*, as the depicted for the chess example [→§1.2], in the form of transfer assessment according to public values (i. e. market values when available, etc.). Alas, we have already shown that fixed rules and any other mechanism that fails to collect signals or information on *transfers*, apart from *sides*, generate inefficient assignments.

5 CONCLUDING REMARKS

A series of problems that consist on allotment of two goods between two individual has been studied in the present work. As it has been shown in the first section of the paper, the types of problems covered in this analysis could be quite broad.

For the linear case examined in the present work, the set of traditional auctions, together with the average-price auction ***[ASB]***, have been found to be efficient among the family of feasible mechanisms. Characterization of *ex ante* efficient rules entails allotment of the *prize* to the higher-typed individual, and thus involves *Side* functions of the form $\tilde{Q}(\tau) = F(\tau)$.

Those positive results on efficient contrast with negative evidence on the existence of efficient rules under general allotment structures[24]. A first extension to the present analysis would be the study of how asymmetry affects the results of this paper.

A second extension to the model would be the analysis of non linear cases (for example,

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[22] That is to say, transfers that do not need any external funds.

[23] Though this result might seem quite trivial, we notice that replacing ***[tDEC]*** for ***[trDEC]*** could have in general more powerful effects, since equivalence between *ex ante* efficiency and maximizing expected aggregate utility is only valid for some mechanism families, but not all.

[24] For instance, Myerson and Satterthwaite (1983) showed in a seminal work the impossibility of existence of *ex post* (globally) efficient, self-sufficient rules for the case of a selling problem under asymmetric valuations between seller and buyer.

concave formulations that deal with risk aversion). However, obtaining equilibria for these instances will in general involve resolution of less tractable equations than the ones for the linear case (for most circumstances, hardly solvable integral [or at best, second-order differential] equations would arise).

It is interesting to note that *direct revelation mechanisms* (as $\mathcal{B}'$) can be built to deal with these bargaining schemes.

Another possible extension, perhaps the most interesting from the economic point of view, though in no way an easy task, is the expansion of the model to cover the problem of assigning *n* goods among *n* individuals.

Overall, this work suggests that auctions, and in general, mechanisms that collect signals for *time* or *transfers* in addition to *sides*, perform better than traditional rules of allotment that fix transfers along with outside, public information. Though the initial, motivational example refers to a chess situation for which rules that collect time bids could yield an efficient result, the same conclusions apply to a wide set of economic settings that can be characterized in the following way: Two goods (*prize* and *residual*) are to be allotted between two individuals, and transfers are designed as a means of compensation for the imbalanced assignment. This framework may comprise an important number of economic problems.

The main conclusion of this article is then that for some allotment settings of two goods between two agents for which a linear form could be assumed, the resulting rules can be analyzed with the classical machinery of Mechanism Design seen for instance in simple bargaining schemes as conventional auctions. Indeed, the rules that possess important efficiency properties in such schemes have been found here to hold those traits. In the end, one is tempted to ask *but another time* why these kinds of mechanisms are not seen more frequently in practice.

MATHEMATICAL APPENDIX

Proof of Lemma 2: Suppose the first proposition applies, but a mechanism $\mathfrak{N}^d \in \mathcal{F}$ *ex post* Pareto-dominates $\mathfrak{M}^d$. First, we set aside analysis over the set of all pairs of points of identical types (τ,τ), since it has null measure over the whole type set. On the other hand, $\mathfrak{N}^d$ must agree with $\mathfrak{M}^d$ on almost all pairs $(\tau_1,\tau_2) \mid \tau_1 \neq \tau_2$, because *ex post* aggregate utility is in these points equal to $\left(\delta_1^W(\tau_1,\tau_2)+\delta_2^W(\tau_1,\tau_2)\right)\cdot(\tau_1-\tau_2)$, expression maximized (under those constrains defining the family $\mathcal{F}$) only when $\delta_l^W(\tau_1,\tau_2)=1 \Leftrightarrow \tau_l=\max(\tau_1,\tau_2)$.

Now let us suppose $\mathfrak{M}^d$ is *ex post* efficient over $\mathcal{F}$, but fails to satisfy the first proposition of the thesis. A contradiction will be found as follows: construct the mechanism $\mathfrak{N}^d$ so that its outcome equals that of $\mathfrak{M}^d$, except for the points $(\tau_1,\tau_2) \mid \tau_1 \neq \tau_2$ for which the first proposition is false. Over this set of points i, of the form (τ_1^i,τ_2^i), let us take without loss of generality $\tau_1^i > \tau_2^i$; we define $\delta=\delta_1^W(\tau_1^i,\tau_2^i)$ and $t=\hat{t}(\tau_1^i,\tau_2^i)$ under $\mathfrak{M}^d$. Since the first proposition of the theorem is not verified in such points i, we may assign $\delta_1^W(\tau_1,\tau_2)$ under $\mathfrak{N}^d$ to be $\delta+\varepsilon<1$, for ε positive but sufficiently small. If $\hat{t}(\tau_1,\tau_2)$ under $\mathfrak{N}^d$ is set to be $t+\dfrac{2\varepsilon(\tau_1-t)}{2\delta-1}$, $\hat{u}_1(\tau_1,\tau_2)$ under this new mechanism is equal to the *ex post* utility of the first player under $\mathfrak{M}^d$; however, for the second individual the mechanism $\mathfrak{N}^d$ assigns *ex post* utility greater than[25] the set under the former rule. Moreover, $\mathfrak{N}^d$ clearly belongs to the family $\mathcal{F}$. Thus, it cannot be true that $\mathfrak{M}^d$ is *ex post* efficient over that family of mechanisms.

QED

[25] In fact, it compares greater by the amount $2\varepsilon(\tau_1-\tau_2)>0$.

Proof of Theorem 3: The equivalence of b) and c) is straightforward, since for both it is inferred that $\widetilde{Q}_l(\cdot)$ is a subderivative of $\widetilde{u}_l(\cdot)$. [In fact, due to the character of $\widetilde{Q}_l(\cdot)$, this function is at worst a one-side (i. e. right- or left-hand) derivate]. A function $u(\tau)$ is convex if and only if its subdifferential is a [convex-valued, upper-hemicontinuous] correspondence $\varphi(\tau)$ which is non-void and non-decreasing [so that $\forall\tau \quad \varphi(\tau)\neq\varnothing$ and $\tau^1 > \tau^2 \Rightarrow \max\varphi(\tau^1)\geq \min\varphi(\tau^2)$].

To prove that a) implies b), consider the previous definition of $\hat{u}_l(\tau_l,t)$. Since $\widetilde{u}_l(\tau_l)=\operatorname{Sup}_t \hat{u}_l(\tau_l,t)$, $\operatorname{epigr}(\widetilde{u}_l(\tau_l))$ [the epigraph of $\widetilde{u}_l(\tau_l)$] is the intersection of the epigraphs of the family of functions $\{\hat{u}_l(\tau_l,t)\}_t$. Thus, for any *t* the function $\hat{u}_l(\tau_l,t)$ must lie below the graph of $\widetilde{u}_l(\tau_l)$. Furthermore, $\operatorname{epigr}(\widetilde{u}_l(\tau_l))=\bigcap_t epigr(\hat{u}_l(\tau_l,t))$ is convex, as is the intersection of half-planes.

From the quasicontinuous character of $\widetilde{Q}_l(\cdot)$, indeed we infer that for $t=\tau_l$ the function $\hat{u}_l(\tau_l,t)$ is tangent to the graph of $\widetilde{u}_l(\tau_l)$ for almost all τ_l. In those points, $\frac{\mathrm{d}\widetilde{u}_l(\tau_l)}{\mathrm{d}\tau_l}=\frac{\mathrm{d}\hat{u}_l(\tau_l,t)}{\mathrm{d}\tau_l}=\widetilde{Q}_l(t)=\widetilde{Q}_l(\tau_l)$. Extending this reasoning to the set of points for which $\widetilde{Q}_l(\cdot)$ is discontinuous, we deduce that $\widetilde{Q}_l(\cdot)$ is [at worst] a one-side derivative for $\widetilde{u}_l(\tau_l)$. Applying the Fundamental Theorem of Calculus over the Riemann-integrable function $\widetilde{Q}_l(\cdot)$ we obtain the equivalence $\widetilde{u}_l(\tau_l)\equiv\int_0^{\tau_l}\widetilde{Q}_l(y)\mathrm{d}y-\widetilde{T}_l(0)$, since $\widetilde{u}_l(0)=-\widetilde{T}_l(0)$.

Finally, to show that b) implies a), we note that the former proposition implies that $\widetilde{Q}_l(y)$ acts as a subderivative (indeed again at worst as a one-side derivative) of $\widetilde{u}_l(y)$ at every point $y\in[0;1]$; hence, the line $\widetilde{Q}_l(y)\cdot\tau_l-\widetilde{T}_l(y)$ is *subtangent* to $\widetilde{u}_l(\cdot)$ at every point y of the unit interval (it coincides with $\widetilde{u}_l$ at y and never lies above the graph of that function). In other words,

$$\forall y\in[0;1]\,\forall\tau_l\in[0;1],\ \widetilde{Q}_l(y)\cdot\tau_l-\widetilde{T}_l(y)\leq\widetilde{u}_l(\tau_l) \qquad \textbf{[26]}$$

Since, by definition, $\widetilde{u}_l(\tau_l)=\widetilde{Q}_l(\tau_l)\cdot\tau_l-\widetilde{T}_l(\tau_l)$, we conclude that

$$\forall\tau_l\ \widetilde{u}_l(\tau_l)=\operatorname*{Sup}_{y\in[0;1]}\widetilde{Q}_l(y)\cdot\tau_l-\widetilde{T}_l(y)$$

QED

Proof of Theorem 4: **1**: Direct implication: For any *l*, Theorem 3 assures that

$$\widetilde{u}_l^{I}(\tau_l)=\int_0^{\tau_l}\widetilde{Q}_l^{I}(y)\mathrm{d}y-\widetilde{T}_l^{I}(0) \qquad \textbf{[27]}$$

and

$$\widetilde{u}_l^{II}(\tau_l)=\int_0^{\tau_l}\widetilde{Q}_l^{II}(y)\mathrm{d}y-\widetilde{T}_l^{II}(0) \qquad \textbf{[28]}$$

Since those results are valid for all τ_l, subtracting term by term and re-arranging the expression [taking into account that $\int_0^{\tau_l}\widetilde{Q}_l^{I}(y)\mathrm{d}y=\int_0^{\tau_l}\widetilde{Q}_l^{II}(y)\mathrm{d}y$], the rightmost identity in the thesis is obtained.

Converse implication: Again **[27]** and **[28]** are valid under Theorem 3. If the rightmost identity in the thesis is valid, then it must be the case that $\forall\tau_l\ \int_0^{\tau_l}\widetilde{Q}_l^{I}(y)\mathrm{d}y=\int_0^{\tau_l}\widetilde{Q}_l^{II}(y)\mathrm{d}y$. This in turn implies the congruence of $\widetilde{Q}_l^{I}(\cdot)$ and $\widetilde{Q}_l^{II}(\cdot)$ but for a set of null measure.

2: The proof of this proposition parallels that of the previous direct implication, except that expressions **[27]** and **[28]** are now integrated over the unit interval.

3: $\breve{u}_1^I = \breve{u}_2^I$ is a trivial consequence of fulfilment of *[ANON]*[26]. Then, expected aggregate utility $\sum_{i=1}^{2} \breve{u}_i^I = 2\breve{u}^I$ equals

$$\int_0^1\int_0^1 \{q_1^I(\tau_1,\tau_2)[\tau_1 - t_1^I(\tau_1,\tau_2)] + q_2^I(\tau_1,\tau_2)[t_1^I(\tau_1,\tau_2) - \tau_1]\} f^I(\tau_1) f^I(\tau_2) \mathrm{d}\tau_2\, \mathrm{d}\tau_1 + \int_0^1\int_0^1 \{q_2^I(\tau_1,\tau_2)[\tau_2 - t_2^I(\tau_1,\tau_2)] + q_1^I(\tau_1,\tau_2)[t_2^I(\tau_1,\tau_2) - \tau_2]\} f^I(\tau_1) f^I(\tau_2) \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \quad \textbf{[29]}$$

$$= \int_0^1\int_0^1 \{q_1^I(\tau_1,\tau_2)[\tau_1 - \tau_2] + q_2^I(\tau_1,\tau_2)[\tau_2 - \tau_1]\} f^I(\tau_1) f^I(\tau_2) \mathrm{d}\tau_2\, \mathrm{d}\tau_1 \quad \textbf{[30]}$$

$$= \int_0^1\int_0^1 \{[2q_1^I(\tau_1,\tau_2) - 1]\tau_1\} f^I(\tau_1) f^I(\tau_2) \mathrm{d}\tau_2\, \mathrm{d}\tau_1 + \int_0^1\int_0^1 \{[2q_2^I(\tau_1,\tau_2) - 1]\tau_2\} f^I(\tau_1) f^I(\tau_2) \mathrm{d}\tau_2\, \mathrm{d}\tau_1 \quad \textbf{[31]}$$

$$= 2\int_0^1 \tau \cdot \widetilde{Q}^I(\tau) f^I(\tau) \mathrm{d}\tau = 2\int_0^1 \left(\widetilde{u}^I(\tau) + \widetilde{T}^I(\tau)\right) f^I(\tau) \mathrm{d}\tau \quad \textbf{[32]}$$

$$= 2\breve{u}^I + 2\int_0^1 \widetilde{T}^I(\tau) f^I(\tau) \mathrm{d}\tau \quad \textbf{[33]}$$

…where some properties of feasible mechanisms have been applied. The expressions of the thesis are straightforward derivations of equations **[32]** and **[33]**.

4: From the results of Theorem 3 and its own definition, $\breve{u}^I = \int_0^1 \widetilde{u}^I(\tau) f^I(\tau) \mathrm{d}\tau$ can be written as:

$$\int_0^1 \left\{ \int_0^\tau \widetilde{Q}^I(y) \mathrm{d}y - \widetilde{T}^I(0) \right\} f^I(\tau) \mathrm{d}\tau = \int_0^1 \left(\int_0^\tau \widetilde{Q}^I(y) \mathrm{d}y \right) f^I(\tau) \mathrm{d}\tau - \widetilde{T}^I(0)$$

Hence, of course,

$$\widetilde{T}^I(0) = \int_0^1 \int_0^\tau \widetilde{Q}^I(y) f^I(\tau) \mathrm{d}y\, \mathrm{d}\tau - \breve{u}^I$$

Since *[ANON]* is fulfilled, we may replace $\breve{u}^I$ by $\int_0^1 \tau \cdot \widetilde{Q}^I(\tau) f^I(\tau) \mathrm{d}\tau$ (in accordance with the third proposition of this theorem) to obtain the first equality of the thesis. The last equality obtains after integrating by parts the previous integral.

QED

<u>*Proof of Theorem 5*</u>: While proving the preceding theorem, it was shown (equation **[30]**) that *ex ante* aggregate expected utility is given by the integral

$$\int_0^1\int_0^1 \{q_1(\tau_1,\tau_2)[\tau_1 - \tau_2] + q_2(\tau_1,\tau_2)[\tau_2 - \tau_1]\} f(\tau_1) f(\tau_2) \mathrm{d}\tau_2\, \mathrm{d}\tau_1$$

Let us consider the set of feasible mechanism $\mathfrak{M}$ which solve the problem

$$\max \sum_{h=1}^{2} \breve{u}_h \quad \textbf{[34]}$$

under all the pertaining restrictions. We know that such mechanisms are *ex ante* feasibly efficient (see Lemma 1).

In particular, under *[qDEC]*, **[34]** could be re-expressed (following equation **[30]**) as

$$\max_{q_1(\cdot), q_2(\cdot)} \int_0^1\int_0^1 \{q_1(\tau_1,\tau_2)[\tau_1 - \tau_2] + q_2(\tau_1,\tau_2)[\tau_2 - \tau_1]\} f(\tau_1) f(\tau_2) \mathrm{d}\tau_2\, \mathrm{d}\tau_1$$

…through the choice of suitable functions $q_l(\cdot)$ so that fulfilment of the applicable constrains is assured. Under *[qDEC]*, functions $q_l(\cdot)$ must take values on the unit interval. Consider then the following problem:

$$\max_{q_1(\cdot), q_2(\cdot)} \int_0^1\int_0^1 \{q_1(\tau_1,\tau_2)[\tau_1 - \tau_2] + q_2(\tau_1,\tau_2)[\tau_2 - \tau_1]\} f(\tau_1) f(\tau_2) \mathrm{d}\tau_2\, \mathrm{d}\tau_1$$

s. t. **[35]**

[26] And the fact that probability distributions are symmetric, which by the way has been assumed throughout the paper.

$\forall l \in \{1;2\}\ \forall \tau_1, \tau_2 \in [0;1]\ \ 0 \le q_l(\tau_l, \tau_{-l}) \le 1$

A solution to **[35]** is given by the pair of functions $q_l(\cdot)$ satisfying

$$q_1(\tau_1, \tau_2) = 1 - q_2(\tau_1, \tau_2) \qquad \textbf{[36]}$$

and

$$q_l(\tau_l, \tau_{-l}) = 1 \Leftrightarrow \tau_l > \tau_{-l} \qquad \textbf{[37]}$$

Let $\overline{U}$ stand for the solution to **[35]**. If the pair $q_l(\cdot)$ satisfy **[36]** and **[37]**, then both *Side* functions $\widetilde{Q}_l(\tau_l)$ equal $F(\tau_l)$.

Any of the auctions generates functions $q_l(\cdot)$ satisfying both **[36]** and **[37]**. Thus, *ex ante* aggregate utility under any of these rules must equal $\overline{U}$, which according to Lemma 1 implies the thesis of the theorem. In fact, any feasible mechanism must allot an *ex ante* aggregate utility $\sum_{h=1}^{2} \breve{u}_h$ not greater than $\overline{U}$: otherwise, the mechanism must violate either **[36]** or **[37]** and hence would not be feasible, since ***[qDEC]*** could not be fulfilled in that case. Thus, the family of auctions cannot be [*ex ante, interim, ex post*] Pareto-dominated by any other feasible mechanism; indeed, as it is claimed in the thesis, auctions cannot be Pareto-dominated by any other rule in $\mathscr{F}$.

QED

<u>*Proof of Theorem 7*</u>*:* Under $\mathscr{B}'$, $\widetilde{Q}(\tau) = F(\tau) = \tau$. Hence, any of the auctions and this mechanism share the same *Side* function. We now study the not so direct formulation of the *Time* function $\widetilde{T}(\tau)$.

We know that $\widetilde{T}(\tau) = \widetilde{Q}(\tau) T^W(\tau) + (\widetilde{Q}(\tau) - 1) T^B(\tau)$. For any $\tau \in \mathscr{S}$, we may define the *binary expansion* of τ as $\alpha(\tau) = (\alpha_h(\tau))_{h=1}^{\infty}$ so that $\forall h = 1,2\ldots\ \ \alpha_h(\tau) \in \{0;1\}$ and $\tau = \sum_{h=1}^{\infty} \alpha_h 2^{-h}$. Now, let's define the function

$$\mathcal{T}(\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_i(\tau) \alpha_j(\tau) 2^{-(i+j)} \qquad \textbf{[38]}$$

It is not difficult to see that $T^W(\tau) = \dfrac{\mathcal{T}(\tau)}{\tau}$, since $\mathcal{T}(\cdot)$ is correlated to the expected time (under τ) to be allotted as white; dividing $\mathcal{T}(\tau)$ by τ we get the required expected time $T^W(\tau)$ [27].

By symmetry, $T^B(\tau) = 1 - \dfrac{\mathcal{T}(1-\tau)}{1-\tau}$.

The following lemma will be useful afterwards.

<u>***Lemma 4***</u>*:* $\forall \tau \in \mathscr{S}$, $\mathcal{T}(\tau) + \mathcal{T}(1-\tau) + \tau \cdot (1-\tau) = 2/3$.

<u>*Proof of Lemma 4*</u>*:* By definition,

$$\mathcal{T}(\tau) + \mathcal{T}(1-\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \alpha_i(\tau) \alpha_j(\tau) 2^{-(i+j)} + \sum_{i=1}^{\infty} \sum_{j=1}^{i} (1-\alpha_i(\tau))(1-\alpha_j(\tau)) 2^{-(i+j)}$$

τ times $1-\tau$ (considered as a binary product) is

$$\tau(1-\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i(\tau) \alpha_j(1-\tau) 2^{-(i+j)}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i(\tau)(1-\alpha_j(\tau)) 2^{-(i+j)}$$

[27] $\mathcal{T}(\tau)$ constitutes some weighted sum of possible times to be allotted if the player of type τ is assigned the white pieces. We then divide this addition by τ , the probability related to the condition that this player is actually allotted the white side.

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\alpha_i(\tau)(1-\alpha_j(\tau))2^{-(i+j)} + \sum_{j=1}^{\infty}\sum_{i=1}^{j}\alpha_i(\tau)(1-\alpha_j(\tau))2^{-(i+j)}$$
$$-\sum_{h=1}^{\infty}\alpha_h(\tau)(1-\alpha_h(\tau))2^{-2h} \qquad \textbf{[39]}$$

Rearranging these formulations, we arrive to

$$\mathcal{T}(\tau)+\mathcal{T}(1-\tau)+\tau(1-\tau)=\sum_{i=1}^{\infty}\sum_{j=1}^{i}2^{-(i+j)}$$
$$=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}2^{-(i+j)}-\sum_{j=1}^{\infty}\sum_{i=1}^{j}2^{-(i+j)}+\sum_{h=1}^{\infty}2^{-2h}$$

Hence,

$$\mathcal{T}(\tau)+\mathcal{T}(1-\tau)+\tau(1-\tau)=\frac{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}2^{-(i+j)}+\sum_{h=1}^{\infty}2^{-2h}}{2}$$
$$=\frac{1+\frac{1}{3}}{2}=\frac{2}{3}$$

QED

Thus, reverting to the proof of the theorem, we write the *Time* function as

$$\tilde{T}(\tau)=\tilde{Q}(\tau)T^{W}(\tau)+(\tilde{Q}(\tau)-1)T^{B}(\tau)=\mathcal{T}(\tau)+\mathcal{T}(1-\tau)+\tau-1 \qquad \textbf{[40]}$$

By Lemma 4, $\tilde{T}(\tau)+1-\tau^2=\frac{2}{3}$.
Rearranging terms, we arrive to

$$\tilde{T}(\tau)=\tau^2-\frac{1}{3} \qquad \textbf{[41]}$$

…which by the way we note is precisely the *Time* function under any of the auctions.
Thus, the *interim* utility under $\mathcal{B}'$…

$$\tilde{u}(\tau)=\tau^2-\tau+\frac{1}{3} \qquad \textbf{[42]}$$

…coincides (for the uniform case) with the *interim* utility under any of the auctions.
Since any of those mechanisms fulfil ***[IC]*** over the domain $[0;1]$, they must satisfy the weaker condition

$$\forall\tau\in\Im \quad \tilde{u}(\tau)=\sup_{t\in\Im}\hat{u}(\tau,t) \qquad \textbf{[43]}$$

Thus, it is evident that $\mathcal{B}'$ must satisfy ***[IC]*** on the domain $\Im$. Otherwise, for some τ the equation **[43]** would be violated.
The functions $t_l(\tau_l,\tau_{-l})$ and $q_l(\tau_l,\tau_{-l})$ generated by $\mathcal{B}'$ are fairly simple to compute; it is straightforward to verify compliance of both conditions ***[tDEC]*** and ***[qDEC]***, as well as ***[mTIME]***, as stated.
Finally, to show that $\mathcal{B}'$ is *ex ante* efficient under domain $\Im$, we note that this mechanism, as well as any of the studied auctions, maximizes an integral almost identical to the one of Theorem 5, except for the integration domain, which is in the present case the "rectangle" $\Im^2$ rather than $[0;1]^2$.

QED

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