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DYNAMIC ANALYSIS OF PLANE MOORING CHAINS OF INEXTENSIBLE LINKS

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Abstract. In this paper the dynamic problem of inextensible chains is addressed. Chains and cables are employed as mooring devices as well as in other structural applications. The dynamic response of structural elements (e.g. floating platforms) joined to the chains/cables are influenced by the strong non-linearity which is of the geometric rather than any material type. The nonlinear Differential-Algebraic Equations (DAE) are derived by direct dynamic equilibrium. The chain may be subjected to general loads. It is also considered that both ends can undergo arbitrary dynamic displacements. The ordinary DAE are tackled by means of temporal power series. It is worthwhile to mention that the explicit expansion in the time variable leads to a linear algebraic system in the series coefficients for each power of t, despite the strong nonlinearity of the system. The consequent advantage is the availability of an analytical solution that allows the validation of other numerical solutions. The algorithm is illustrated by numerical examples in which the chain is subjected to self-weight with one end fixed and the other in prescribed motion. The different trajectories of the chain dynamic response are presented. Any number of links may be considered and taking a large number of links gives place to an inextensible cable model.

1 INTRODUCTION

Slack cables and chains are used as mooring devices as well as for other structural applications and they act coupled with elements constituting dynamic systems that undergo nonlinear motions. The main source of the nonlinearity from the cables and chains is the inherent geometric nonlinearity (Esmailzadeh and Goodarzi, 2001; Rosales and Filipich, 2006). Interesting and detailed work on cables and chains is included in two theses (Tibert, 1999; Gobat, 2000) in which different approaches are dealt with including analytical and experimental studies. From the many works published in this subjects some may be cited as related to the present study (Huang, 1994; Liu and Bergdahl, 1997; Sannasiraj et al., 1998; Pascoal et al., 2005). The authors have worked in dynamics of cables using a quasi-static model for the cables (Rosales et al., 2003; Escalante et al., 2005; Rosales and Filipich, 2006) with different complexities. In this work the strongly nonlinear motion in a plane of a chain of an arbitrary number N of links is addressed. The chain is assumed inextensible and the governing equations result in an differential-algebraic system of equations (DAE's) which is derived by direct dynamic equilibrium. The chain may be subjected to general loads and it is also considered that both ends can undergo arbitrary dynamic displacements. The ordinary DAE are tackled by means of temporal power series. It is worthwhile to mention that the explicit expansion in the time variable leads to a linear algebraic system in the series coefficients for each power of t, despite the strong nonlinearity of the system. The consequent advantage is the availability of an analytical solution that allows the validation of other numerical solutions. Previous works by the authors applied this approach to diverse strongly nonlinear problems ((Filipich et al., 2004; Rosales and Filipich, 2006)). A distinctive feature of the application of this approach to the dynamics of a chain is that only $(M+1)$ *linear* systems in $4N$ unknowns, where M is the number of terms of the power series, yield. That is, a $(4N \cdot 4N)$ linear system is to be solved for each power of time. After the statement and solution some illustrative examples of the chain dynamics with prescribed displacements at the end are shown and compared with a finite element solution. Finally, the automatization of a classical tool as the integer power series lead to a simple solution to a problem that is basically complex. Additionally, taking a large number of links gives place to an inextensible cable model.

2 PROBLEM STATEMENT

Figure 1 shows an scheme of a chain made of N inextensible links each of length a and uniform section Ω, i.e. the chain total length is $L = Na$. There will be $(N+1)$ hinged nodes and N links. In general, the chain will be subjected, besides its self-weight, to arbitrary conservative forces. The motion will give place to inertial forces and reactions at each node. As mentioned before, the focus will be on dynamic problems with prescribed end displacements, i.e. $x_1(t)$, $y_1(t)$, $x_{N+1}(t)$ and $y_{N+1}(t)$ are input. A free body diagram of a generic portion k (a link) is sketched in Figure 2 in which the coordinates and both active and passive forces and their directions are depicted. As may be observed geometric continuity is accepted at coincident nodes of consecutive links. It may be also observed that, during the motion, the instantaneous resulting force is null. This is due that it is implicitly assumed that no forces are applied at each node. However, this effect may be eventually introduced without difficulty. At the same time, the forces H_1, V_1, H_{N+1} and V_{N+1} will arise from the prescribed displacements. The length element of the link $ds = dX$ which is constant in time since the inextensibility assumption holds. If we denote ρ as the uniform density for all links. Then the mass element dm is

$$dm = \rho\Omega ds = \rho\Omega dX \tag{1}$$

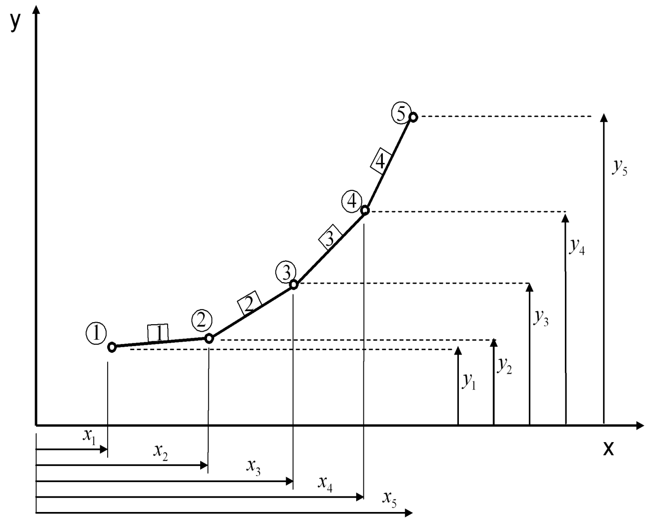

Figure 1: Coordinates of a chain of five links.

and $\gamma = \rho g$ is the unit weight and g is the gravity acceleration. The orthogonal cartesian axes XY are adopted as the inertial reference. The active forces per unit length, variables in general with the link length, have the cartesian components p_k^* and q_k; the vertical component writes

$$p_k^* = \gamma\Omega + p_k \tag{2}$$

The resulting forces are

$$Q_k = \int_0^a q_k ds \tag{3a}$$

$$P_k^* = P + P_k \qquad \text{with} \qquad P \equiv \int_0^a \gamma\Omega ds = \gamma\Omega a \qquad \text{and} \qquad P_k \equiv \int_0^a p_k ds \tag{3b}$$

In order to calculate the inertial resulting forces let us first write the coordinates of a generic point in motion,

$$x = (x_{k+1} - x_k)\left(\frac{X - X_k}{a}\right) + x_k \tag{4a}$$

$$y = (y_{k+1} - y_k)\left(\frac{X - X_k}{a}\right) + y_k \tag{4b}$$

If we denote the time derivatives with dots the accelerations of the generic point are

$$\ddot{x} = (\ddot{x}_{k+1} - \ddot{x}_k)\left(\frac{X - X_k}{a}\right) + \ddot{x}_k \tag{5a}$$

$$\ddot{y} = (\ddot{y}_{k+1} - \ddot{y}_k)\left(\frac{X - X_k}{a}\right) + \ddot{y}_k \tag{5b}$$

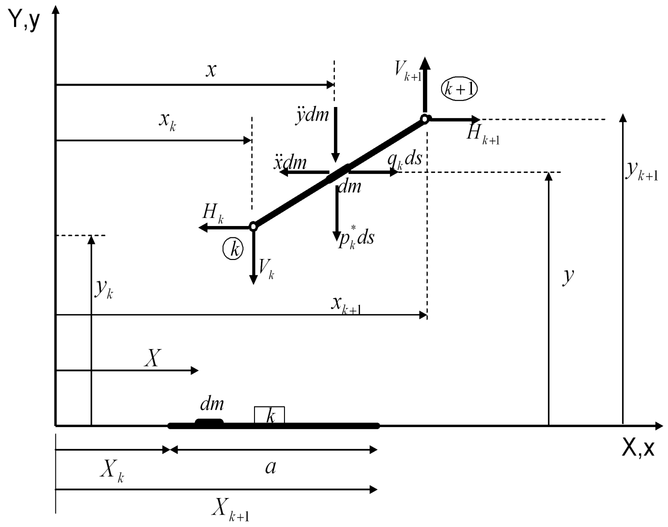

Figure 2: Free forces diagram of a generic link

If the inertial resultants F_X and F_Y are defined as

$$F_{kX} = \int_0^a \ddot{x}dm \qquad F_{kY} = \int_0^a \ddot{y}dm \tag{6}$$

and $R \equiv P/g = \rho\Omega a$ is introduced, the inertial forces result

$$F_{kX} = \frac{R}{2}(\ddot{x}_{k+1} + \ddot{x}_k) \qquad F_{kY} = \frac{R}{2}(\ddot{y}_{k+1} + \ddot{y}_k) \tag{7}$$

It is also necessary to calculate the moment of the forces with respect to node k (or any other arbitrary point of the XY plane). Then the inertial moments (see Fig.2

$$M_k^i = -\int_0^a \left[\ddot{x}\left(y - y_k\right) - \ddot{y}\left(x - x_k\right)\right] dm \tag{8}$$

or after integration and making use of Eqs. (4-5),

$$M_k^i = \frac{R}{6}\left[\left(2\ddot{y}_{k+1} + \ddot{y}_k\right)\left(x_{k+1} - x_k\right) - \left(2\ddot{x}_{k+1} + \ddot{x}_k\right)\left(y_{k+1} - y_k\right)\right] \tag{9}$$

The moment of q_k and p_k^* with respect to node k will be denoted by M_k^* and writes

$$M_k^* = \int_0^a \left[q_k\left(y - y_k\right) - p_k^*\left(x - x_k\right)\right] ds = \frac{P}{2}\left(x_{k+1} - x_k\right) + M_k \tag{10a}$$

where $$M_k = \int_0^a \left[q_k\left(y - y_k\right) - p_k\left(x - x_k\right)\right] ds \tag{10b}$$

Finally the moments due to H_{k+1} and V_{k+1} with respect to node k obviously write:

$$M_k^{HV} = H_{k+1}\left(y_{k+1} - y_k\right) - V_{k+1}\left(x_{k+1} - x_k\right) \tag{11}$$

2.1 Equations of motion

After accepting that the motion develops in plane XY, the three equations that should simultaneously verify are the Newton's equations for a rigid body, in this case the rigid link

$$H_{k+1} - H_k - F_{kX} + Q_X = 0 \tag{12a}$$

$$V_{k+1} - V_k - F_{kY} - P_k - P = 0 \tag{12b}$$

$$M_k^{HV} + M_k^* + M_k^i = 0 \tag{12c}$$

which after introducing Eqs (7,9,10a and 11), may be re-written as follows

$$H_{k+1} - H_k - \frac{R}{2}\left(\ddot{x}_{k+1} + \ddot{x}_k\right) = -Q_k \tag{13a}$$

$$V_{k+1} - V_k - \frac{R}{2}\left(\ddot{y}_{k+1} + \ddot{y}_k\right) = P + P_k \tag{13b}$$

$$\begin{aligned}&\left[H_{k+1} - \frac{R}{6}\left(2\ddot{x}_{k+1} + \ddot{x}_k\right)\right]\left(y_{k+1} - y_k\right) - \\ &\left[V_{k+1} - \frac{R}{6}\left(2\ddot{y}_{k+1} + \ddot{y}_k\right)\right]\left(x_{k+1} - x_k\right) = -\frac{P}{2}\left(x_{k+1} - x_k\right) - M_k\end{aligned} \tag{13c}$$

$$(k = 1, 2, \cdots, N)$$

The governing system is composed of $3N$ equations of motion but as functions of $4N$ functions of time: x_k, y_k $(k = 2, 3, \cdots, N)$ and H_k, V_k $(k = 1, 2, \cdots, N+1)$. Yet other N equations are needed to have a determined system. They arise from the inextensibility condition stated for each link,

$$\begin{aligned}\left(x_{k+1} - x_k\right)^2 + \left(y_{k+1} - y_k\right)^2 &= a^2 \\ (k = 1, 2, \cdots, N)\end{aligned} \tag{14}$$

Equations (13) and (14) constitute a non-linear differential-algebraic system of equations (DAE's) of $4N$ equations with $4N$ unknowns that governs the dynamics of a chain of N links pinned among them and with prescribed end motions.

2.2 Initial conditions (IC)

The IC must be imposed to the coordinates involved in the DAE's; in our case $x_k(t)$, $y_k(t)$ $(k = 1, 2, \cdots, N+1)$ govern the instant configuration. That is, $x_k(0), y_k(0), \dot{x}_k(0), \dot{y}_k(0)$ $(k = 1, 2, \cdots, N+1)$ must be known or imposed. Let us introduced the notation $x_{k0} \equiv x_k(0)$, $y_{k0} \equiv y_k(0)$, $\dot{x}_{k0} \equiv \dot{x}_k(0)$, $\dot{y}_{k0} \equiv \dot{y}_k(0)$. As mentioned before, x_{10}, y_{10}, y_{10} and $y_{(N+1)0}$ are given data. Conditions (14) should verify at $t = 0$, i.e.

$$\begin{aligned}\left(x_{(k+1)0} - x_{k0}\right)^2 + \left(y_{(k+1)0} - y_{k0}\right)^2 &= a^2 \\ (k = 1, 2, \cdots, N)\end{aligned} \tag{15}$$

It is easy to deduce that the *quasi*-arbitrary selection among the $(N-2)$ x_{k0} and/or y_{k0} with ($k =$ 2 or 3 or 4, $\cdots$, or $(N-1)$) will suffice. In effect, e.g. choosing x_{N0}, $x_{(N-1)0}$, $x_{(N-2)0}$,$\cdots$, x_{30} of equations (15) and y_{N0}, $y_{(N-1)0}$, $y_{(N-2)0}$,$\cdots$, y_{30} are deduced respectively. A 2 x 2 nonlinear system results (for $k = 1$ and $k = 2$ in equations (15) stated in terms of x_{20} and y_{20}. After solved, the initial configuration adapted to nodes 1 and $(N+1)$ and which verifies the N equations (15), yields. Evidently the selection may be combined for x_{k0} and y_{k0} given they sum $(N-2)$ values. Other possibility, and the one herein chosen for the sake of illustration, is to take as initial configuration the static solution of the same chain hanging from ($x_1(0), y_1(0)$) and $(x_{(N+1)}(0), y_{(N+1)}(0))$ under its self-weight. Let us now discuss the initial velocities ($\dot{x}_{k0}, \dot{y}_{k0}$). To obtain them Eqs. (14) are derived once with respect to time. For instant $t = 0$ the following condition should verify

$$\left(x_{(k+1)0} - x_{k0}\right)\left(\dot{x}_{(k+1)0} - \dot{x}_{k0}\right) + \left(y_{(k+1)0} - y_{k0}\right)\left(\dot{y}_{(k+1)0} - \dot{y}_{k0}\right) = 0 \qquad (k = 1, 2, \cdots , N) \tag{16}$$

Recall that, at first, $\dot{x}_{10}, \dot{y}_{10}, \dot{x}_{(N+1)0}$ and $\dot{y}_{(N+1)0}$ are known. Analogously to the initial configuration, $(N-2)$ values of velocities $\dot{x}_{k0}, \dot{y}_{k0}$ or combinations of them for different k's may be chosen with quite freedom. From the fulfillment of Eq. (16) the velocities at other points may be obtained. Finally we now have a linear system in, for instance, $\dot{x}_{20}, \dot{y}_{20}$ that are obtained from the first two equations. In order to conclude with this issue let us say that the *quasi*-arbitrariness above-mentioned to impose the initial $(N-2)$ coordinates and $(N-2)$ velocities, and since we are dealing with a nonlinear problem, is conditioned by two concepts: one is that each addend of condition (15) can not be larger than a^2 and the other is the following. As is known, the convergence ratio of the series to be employed are IC dependent (unlike linear problems). That is, when the solution is needed for times beyond the convergence ratio —analytical continuation— the IC influence will be determinant and, a strong difficulty to correctly determine the forced vibration problem.

3 POWER SERIES. SYSTEMATIZATION

3.1 Statement of the series

As mentioned in the previous subsection 2.1, the $4N$ unknowns of the above proposition are the following temporal functions:

$$x_k(t), y_k(t) \qquad (k = 2, 3, \cdots , N) \tag{17a}$$

$$H_k(t), V_k(t) \qquad (k = 1, 2, \cdots , N+1) \tag{17b}$$

and clearly, the DAE's require of the solution of the non linear system given by the $4N$ Eqs. (13-14). As is foreseen that analytical continuation will be needed to obtain the response in sufficiently long periods of time, the time is divided in temporal intervals of interest. Let the interval of interest be $\Delta t = t_f - t_i$ (there is no loss of generality if we assume $t_i = 0$). It will be in turn, divided in N_p intervals. In this work they are assumed equal which would lead to a loss of numerical precision, though helping to fix ideas and computational simplicity. Eventually each subinterval might be different depending on the convergence ratio in each interval when the solution is continued (adaptive interval). If T is the duration of each subinterval, and with $t_i = 0$

$$T = \frac{\Delta t}{N_p} = \frac{t_f}{N_p} \tag{18}$$

Let us define the unitary temporal variable τ_p for each subinterval as

$$t = T(\tau_p + p - 1) \qquad (p = 1, 2, \cdots, N_p) \tag{19}$$

where the following verifies

$$(p-1)T \leq t \leq pT \tag{20a}$$

$$0 \leq \tau_p \leq 1 \tag{20b}$$

$$(p = 1, 2, \cdots, N_p)$$

In what follows, as long as no confusion arises, the subscript p from τ_p is omitted, i.e. $\tau \equiv \tau_p$. Let $f = \hat{f}(t)$ be some function of time, then

$$f = f(t) = f[T(\tau + p - 1)] = f_p(\tau) \tag{21}$$

In each subinterval of duration T our unknowns (see Eq. (17)) are functions of the unitary variable τ. Accepting the existence of the DAE's solution, the following expansions are proposed:

$$x_{pk}(\tau) = \sum_{j=0}^{\infty} A_{pkj}\tau^j \tag{22a}$$

$$y_{pk}(\tau) = \sum_{j=0}^{\infty} B_{pkj}\tau^j \tag{22b}$$

$$H_{pk}(\tau) = \sum_{j=0}^{\infty} C_{pkj}\tau^j \tag{22c}$$

$$V_{pk}(\tau) = \sum_{j=0}^{\infty} D_{pkj}\tau^j \tag{22d}$$

$$(k = 1, 2, \cdots, N+1)(p = 1, 2, \cdots, N_p)$$

which in turn we assume convergent. By hypothesis the following coefficients

$$A_{p1j}, B_{p1j}, A_{p(N+1)j}, B_{p(N+1)j} \qquad (p = 1, 2, \cdots, N_p)(j = 0, 1, 2, \cdots) \tag{23}$$

are known given the prescribed motion at the extreme nodes of the chain. Since the IC are also data, the next coefficients are also known:

$$A_{1k0}, A_{1k1}, B_{1k0}, B_{1k1} \qquad (k = 2, 3, \cdots, N) \tag{24}$$

After introducing the notation $(\cdot)' \equiv d(\cdot)/d\tau; \quad (\cdot)'' \equiv d^2(\cdot)/d\tau^2$ it is true that $\dot{(\cdot)} = (\cdot)'/T; \quad \ddot{(\cdot)} = (\cdot)''/T^2$. In Eqs. (13) we admit that P, Q_k, P_k and M_k are also analytical, that is the following series are known too

$$P_p(\tau) = P\sum_{j=0}^{\infty} \delta_{j0}\tau^j \tag{25a}$$

$$Q_{pk}(\tau) = \sum_{j=0}^{\infty} Q_{pkj}\tau^j \tag{25b}$$

$$P_{pk}(\tau) = \sum_{j=0}^{\infty} P_{pkj}\tau^j \tag{25c}$$

$$M_{pk}(\tau) = \sum_{j=0}^{\infty} M_{pkj}\tau^j \tag{25d}$$

$$(k = 1, 2, \cdots, N)(p = 1, 2, \cdots, N_p)$$

where the δ_{j0} is the Kronecker delta.

3.2 Systematization of the series

In order to fix ideas, let $F = F(\tau)$ and $G = G(\tau)$ two analytical functions of τ which expansions are

$$F = \sum_{j=0}^{\infty} f_j\tau^j \tag{26a}$$

$$G = \sum_{j=0}^{\infty} g_j\tau^j \tag{26b}$$

3.2.1 Derivation of the series

After introducing the matrix

$$\varphi_{mi} \equiv (i+1)(i+2)\cdots(i+m) = \frac{(m+i)!}{i!} \tag{27}$$

the following statement derives

$$\frac{d^n F}{d\tau^n} = F^{(n)} = \sum_{j=0}^{\infty} \varphi_{nj} f_{(j+n)}\tau^n \qquad (n \geq 1) \tag{28}$$

For $n = 1$ and $n = 2$, $F^{(1)} \equiv F'$ and $F^{(2)} \equiv F''$, respectively.

3.2.2 Product of the series

Let $E = E(\tau)$ be an analytical function such that

$$E = \sum_{j=0}^{\infty} e_j\tau^j \tag{29}$$

If E is in turn, the product of two functions F and G, see Eqs. (26)

$$E = F(\tau)G(\tau) \tag{30}$$

the following relationship holds

$$e_j = \sum_{r=0}^{j} f_r g_{j-r} = \sum_{r=0}^{j} g_r f_{j-r} \qquad (j = 0, 1, 1, \cdots) \tag{31}$$

Expressions (31) are frequently named *Cauchy products*.

3.3 The DAE's for each subinterval

Equations (13-14) that constitute the DAE's are now re-written for each pth subinterval having into account definition (21) and derivative notations,

$$T^2(H_{p(k+1)} - H_{pk}) - \frac{R}{2}(x''_{p(k+1)} + x''_{pk}) = -T^2 Q_{pk} \tag{32a}$$

$$T^2(V_{p(k+1)} - V_{pk}) - \frac{R}{2}(y''_{p(k+1)} + y''_{pk}) = T^2(P_p + P_{pk}) \tag{32b}$$

$$\begin{aligned} &[T^2 H_{p(k+1)} - \frac{R}{6}(2x''_{p(k+1)} + x''_{pk})](y_{p(k+1)} - y_{pk}) - \\ &-[T^2 V_{p(k+1)} - \frac{R}{6}(2y''_{p(k+1)} + y''_{pk})](x_{p(k+1)} - x_{pk}) \\ &= -T^2\left[\frac{P_p}{2}(x_{p(k+1)} - x_{pk}) + M_{pk}\right] \end{aligned} \tag{32c}$$

$$(k = 1, 2, \cdots, N)(p = 1, 2, \cdots, N_p)$$

$$(x_{p(k+1)} - x_{pk})^2 + (y_{p(k+1)} - y_{pk})^2 = a^2 \tag{32d}$$

4 SOLUTION BY MEANS OF POWER SERIES

In order to address the solution of this strongly nonlinear problem and observing the Eqs. (32), the following new series are introduced $(k = 1, 2, \cdots, N)(p = 1, 2, \cdots, N_p)$

$$H_{p(k+1)}(y_{p(k+1)} - y_{pk}) = \sum_{j=0}^{\infty} \sigma_{pkj}\tau^j \tag{33a}$$

$$V_{p(k+1)}(x_{p(k+1)} - x_{pk}) = \sum_{j=0}^{\infty} \gamma_{pkj}\tau^j \tag{33b}$$

$$(2x''_{p(k+1)} + x''_{pk})(y_{p(k+1)} - y_{pk}) = \sum_{j=0}^{\infty} \mu_{pkj}\tau^j \tag{33c}$$

$$(2y''_{p(k+1)} + y''_{pk})(x_{p(k+1)} - x_{pk}) = \sum_{j=0}^{\infty} \nu_{pkj}\tau^j \tag{33d}$$

$$(x_{p(k+1)} - x_{pk})^2 = \sum_{j=0}^{\infty} \alpha_{pkj}\tau^j \tag{33e}$$

$$(y_{p(k+1)} - y_{pk})^2 = \sum_{j=0}^{\infty} \beta_{pkj}\tau^j \tag{33f}$$

$$a^2 = a^2\sum_{j=0}^{\infty} \delta_{j0}\tau^j \tag{33g}$$

where the coefficients are

$$\sigma_{pkj} = \sum_{r=0}^{j} C_{p(k+1)r}[B_{p(k+1)(j-r)} - B_{pk(j-r)}] \tag{34a}$$

$$\gamma_{pkj} = \sum_{r=0}^{j} D_{p(k+1)r}[A_{p(k+1)(j-r)} - A_{pk(j-r)}] \tag{34b}$$

$$\mu_{pkj} = \sum_{r=0}^{j} \varphi_{2r}[2A_{p(k+1)(r+2)} + A_{pk(r+2)}][B_{p(k+1)(j-r)} - B_{pk(j-r)}] \tag{34c}$$

$$\nu_{pkj} = \sum_{r=0}^{j} \varphi_{2r}[2B_{p(k+1)(r+2)} + B_{pk(r+2)}][A_{p(k+1)(j-r)} - A_{pk(j-r)}] \tag{34d}$$

$$\alpha_{pkj} = \sum_{r=0}^{j}[A_{p(k+1)r} - A_{pkr}][A_{p(k+1)(j-r)} - A_{pk(j-r)}] \tag{34e}$$

$$\beta_{pkj} = \sum_{r=0}^{j}[B_{p(k+1)r} - B_{pkr}][B_{p(k+1)(j-r)} - B_{pk(j-r)}] \tag{34f}$$

$$(k = 1, 2, \cdots, N)(p = 1, 2, \cdots, N_p)(j = 2, 3, \cdots)$$

Once the power series are introduced in the DAE's (32) and given the linear independence of each power $\tau^j (j = 0, 1, \cdots)$ the next equations should verify

$$T^2(C_{p(k+1)j} - C_{pkj}) - \frac{R}{2}\varphi_{2j}(A_{p(k+1)(j+2)} + A_{pk(j+2)}) = -T^2 Q_{pkj} \tag{35a}$$

$$T^2(D_{p(k+1)j} - D_{pkj}) - \frac{R}{2}\varphi_{2j}(B_{p(k+1)(j+2)} + B_{pk(j+2)}) = T^2(P\delta_{j0} + P_{pkj}) \tag{35b}$$

$$(T^2\sigma_{pkj} - \frac{R}{6}\mu_{pkj}) - (T^2\gamma_{pkj} - \frac{R}{6}\nu_{pkj}) = -T^2[P(A_{p(k+1)j} - A_{pkj}) + M_{pkj}] \tag{35c}$$

$$\alpha_{pkj} + \beta_{pkj} = a^2\delta_{j0} \tag{35d}$$

$$(k = 1, 2, \cdots, N)(p = 1, 2, \cdots, N_p)(j = 2, 3, \cdots)$$

Equations (35) give place *prima facie* to the solution of the dynamic of the chain of N links. However two issues have to be tackled. One is the fact that since the IC are known and the motion prescribed at the ends, not all the coefficients in these equations are unknowns. The other issue is that, when using this systematization of the power series a linear system of order $4N$ has to be solved for each j. An increment in the value of j, leads to a change in the matrix of the $4N$ unknowns and the $4N$ independent terms are also modified as function of the unknowns already solved for j less than the one fixed.

Finally, the unknowns to be solved for $(j = 0, 1, 2, \cdots, N + 1)$ are

$$A_{pk(j+2)}, B_{pk(j+2)} \tag{36a}$$

Table 1: Coordinates of initial geometry of the chain (static solution under self-weight)

Coordinate	NODE										
	1	2	3	4	5	6	7	8	9	10	11
x_0	0	4.45	9.13	13.72	18.02	21.82	25.14	28.08	30.66	32.95	35
y_0	0	-1.52	-1.87	-0.98	0.98	3.74	7.05	10.73	14.67	18.77	23

$$(k = 2, 3, \cdots, N)$$

$$C_{pkj}, D_{pkj} \tag{36b}$$

$$(k = 1, 2, \cdots, N + 1)$$

On the other hand, Eqs. (35a to 35c) always involve the unknowns $A_{pk(j+2)}, B_{pk(j+2)}$, $(j \geq 0)$, while Eq. (35d) deals with A_{pkj}, B_{pkj}, $(j \geq 0)$. This means that the index j in Eq. Eq. (35d) should be shifted as follows

$$\alpha_{pk(j+2)} + \beta_{pk(j+2)} = 0 \tag{37}$$
$$(k = 1, 2, \cdots, N)(p = 1, 2, \cdots, N_p)$$

Let us say that Eqs. (35d) for $j = 0, 1$ are verified by IC and prescribed motions at the ends.

5 NUMERICAL ILLUSTRATION

The dynamic response of a chain of inextensible links was analyzed by means of the above-proposed power series approach. The total length of the chain is $L = 47$ m. The magnitudes $A_q = 30$ m and $A_h = 20$ m. The linear density was assumed $\gamma\Omega = 50$ N/m. The chain was supposed fixed at the left end with its right end subjected to prescribed motions with the following cosine functions

$$x_{11}(t) = A_q + q_0 \cos \omega_q t \qquad y_{11}(t) = A_h + h_0 \cos \omega_h t \tag{38}$$

Different values of the initial values q_0 and h_0, as well as the frequencies $\omega_q = r_q\omega$ and $\omega_h = r_h\omega$ of the prescribed displacements were used. However in all the cases, the initial condition IC of the chain was adopted to be the static solution of the chain subjected to self-weight (also solved by the authors with a power series approach, though not shown here for brevity). Also a comparison was performed using the finite element commercial code ALGOR. The first illustration deals with a chain of ten links ($N = 10$), and each link of length of $a = 4.7$ m, $\omega = 0.25$ rad/s, $r_q = 2$, $r_h = 5$, $q_0 = 5$ m, $h_0 = 3$ m, $M = 5$. The initial geometry given by the catenary of the static solution under self-weight, is depicted in Table 1 and in Figure 3 (All coordinates are in meters)

The motion trajectory of the right end of the chain is shown in Figure 4 for 10 s. Figure 5 graphics the motion of the chain by superposing different positions, for a total time of 10 s. Colors vary from blue at start time to orange an the end of the interval. Also the right-end node is shown with small blue circles (see Fig.4). In this case $N_p = 800$ and $\Delta t = 0.0125$ s.

As mentioned above, one of the aims of this work is to find the forces at the right-end as functions of the motions q and h so as to couple this with another body and study the dynamics of the whole system. Here, a prescribed motion at the right-end is given in order to valid the

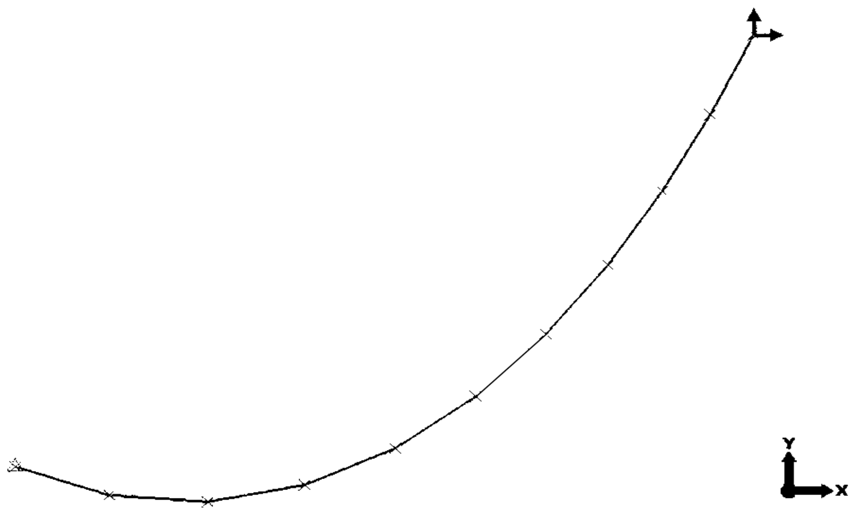

Figure 3: Illustration examples 1 and 2. Initial configuration for the chain (Nodes coordinates depicted in Table 1)

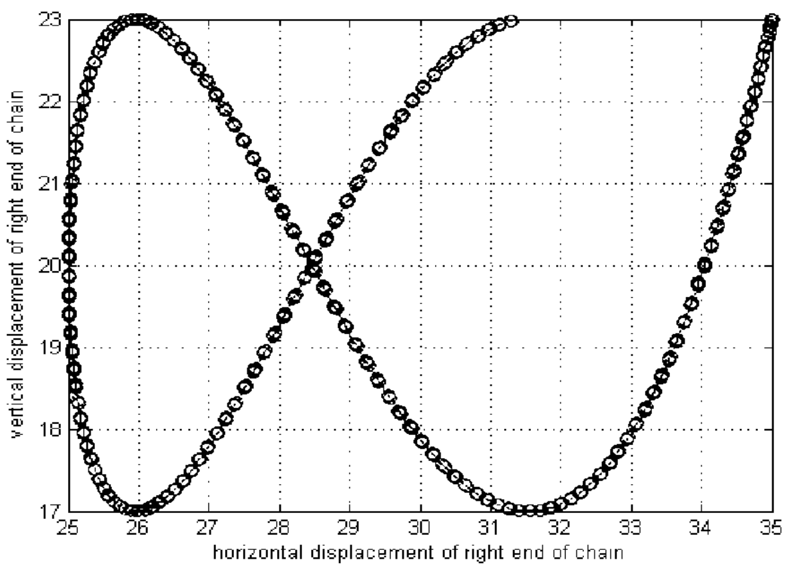

Figure 4: Illustration example 1. Trajectory of right-end node (Prescribed motion).

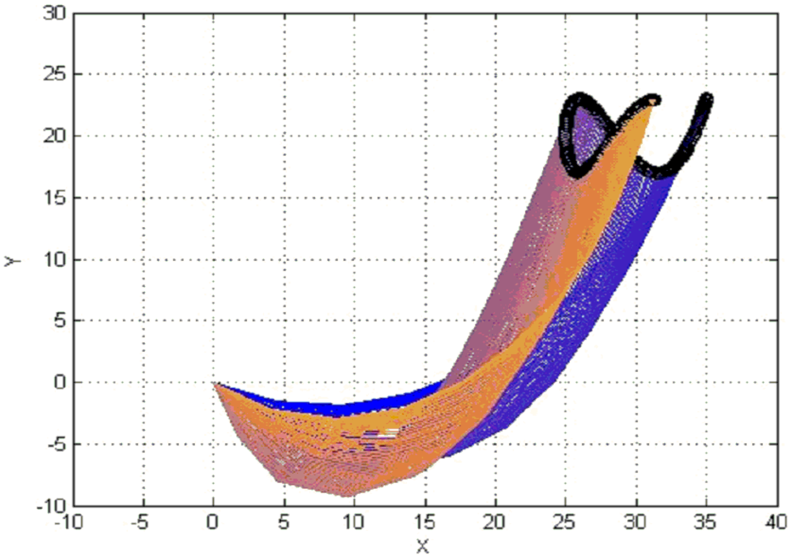

Figure 5: Illustration example 1. Chain motion at different times. Total time 10 s. (As time increases, color changes from blue to orange). Small circles indicate the right-end motion.

algorithm and the left end is kept fixed. The resulting forces for the present example were also found for the interval under study and their variation is shown in Fig. 6 (All forces are in Newtons).

Finally a second example was made for the case of $\omega = 0.25$ rad/s, $r_q = 1$, $r_h = 9$, $q_0 = 5$ m, $h_0 = 3$ m, $N_p = 400$ and $\Delta t = 0.005$ s. The total interval of time is 2 s. The other data were assumed as in illustration example 1. The illustration example 2 was solved with the present power series algorithm (results are shown in Figs. 7 and 8) and with a finite element commercial code. Figure 7 depicts the horizontal and vertical motions at node four. The resulting forces at the right-end of the chain is plotted in Fig. 8.

For comparison a truss model was solved with ALGOR, a commercial finite element code. Each link was modeled as a truss element. The left-end was assumed fixed and the motion at the right-end was given through *prescribed displacements* with the curves shown in Figure 9 that show the variation of the multiplier of the horizontal and vertical displacements which in turn were set with modulus $q_0 = 5$ m, $h_0 = 3$ m, respectively. The time-displacement curves corresponding to the fourth node are included in Fig. 10. A reasonable agreement can be observed from the comparison with values found with the power series algorithm (Fig. 7). Additionally the axial forces at node four, found with the finite element code are depicted in Fig. 11.

6 FINAL COMMENTS

The dynamics of a slack inextensible chain subjected to conservative loads was stated and solved through a power series approach. The strongly nonlinear behavior is governed by a differential-algebraic system of equations (DAE's). If the chain is made of N links, the systematization of the power series allows to the statement of only $(M + 1)$ *linear* systems in $4N$ unknowns, where M is the number of terms of the power series, yield. That is, a $(4N \cdot 4N)$ linear system is to be solved for each power of time. Two examples were presented, in both cases with prescribed motions at the right end. The displacement at each node, the axial forces

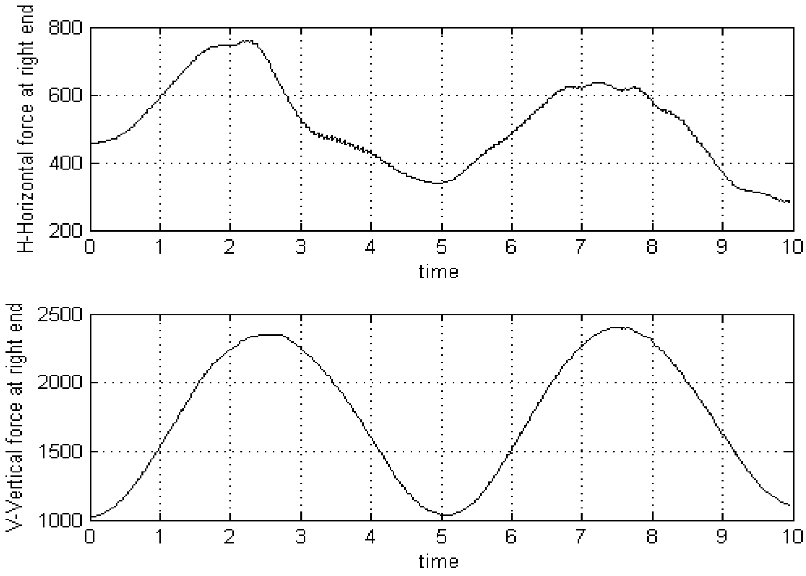

Figure 6: Illustration example 1. Horizontal and vertical forces at right-end of chain.

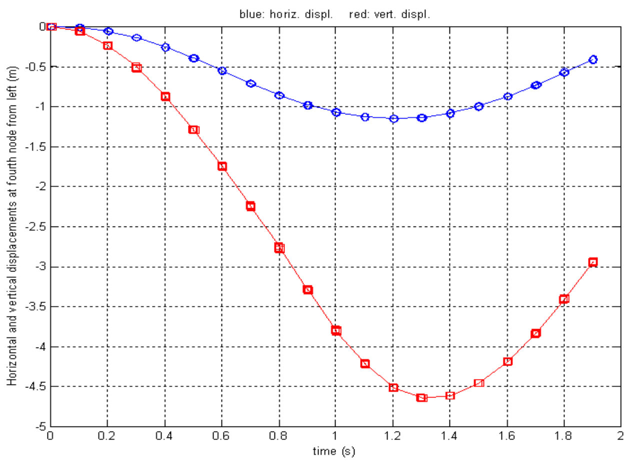

Figure 7: Illustration example 2. Horizontal and vertical displacement at the fourth node. Results found with power series algorithm (present work).

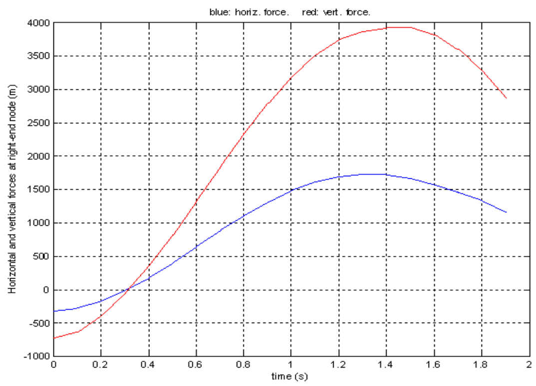

Figure 8: Illustration example 2. Horizontal and vertical forces at right-end node. Results found with power series algorithm (present work).

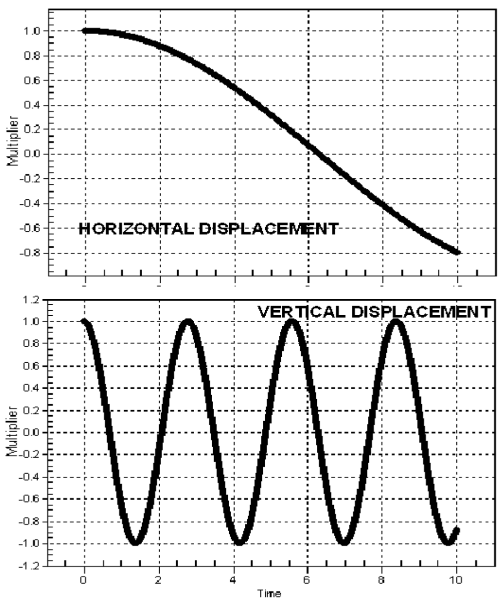

Figure 9: Illustration example 2. Multiplier curves for the prescribed displacements at right end (FEM solution).

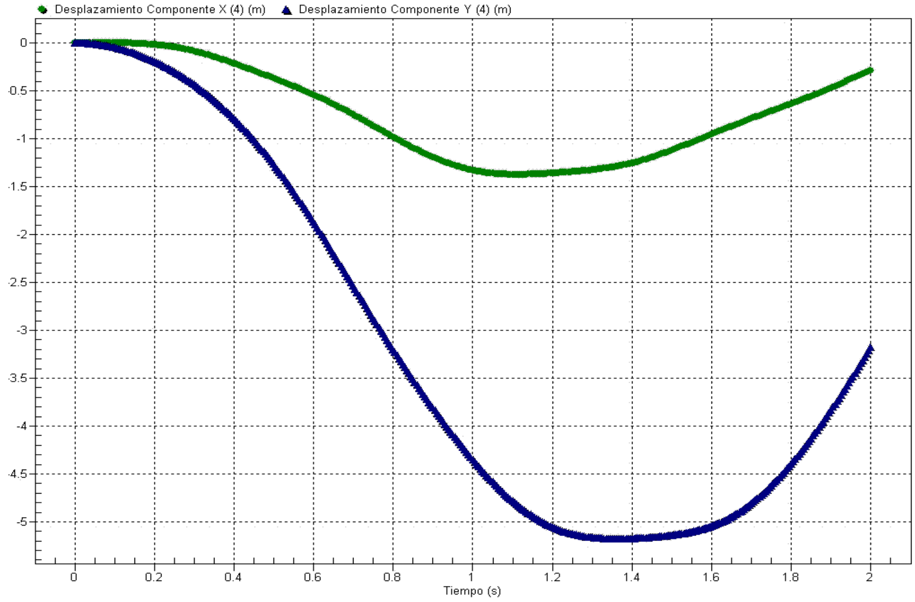

Figure 10: Illustration example 2. Horizontal and vertical displacement at the fourth node.(FEM solution).

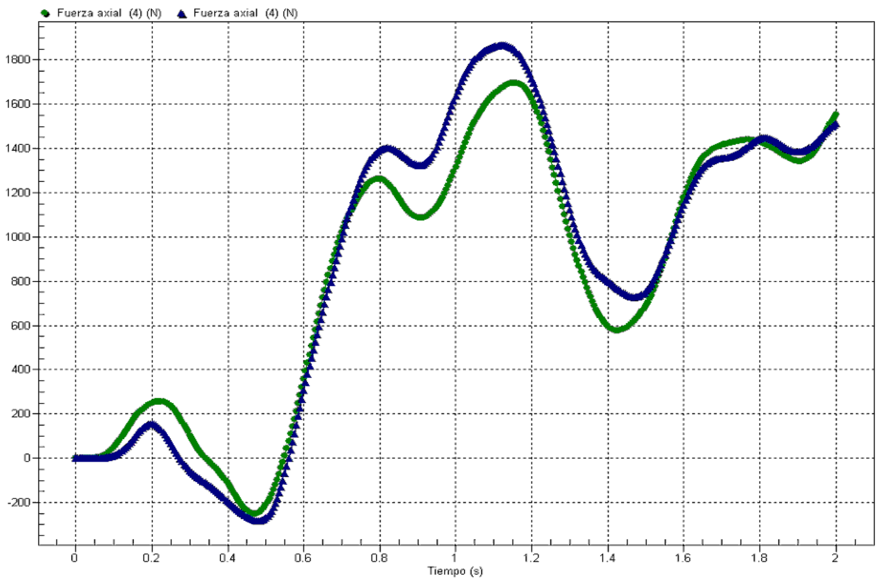

Figure 11: Illustration example 2. Axial forces at the fourth node.(FEM solution).

and in particular the end forces may be calculated as function of time. A comparison with a similar model analyzed with a finite element code shows acceptable agreement.

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