Mecánica Computacional Vol. XXI, pp. 2025–2039
S. R. Idelsohn, V. E. Sonzogni and A. Cardona (Eds.)
Santa Fe–Paraná, Argentina, October 2002

EFFECTS OF PARASITIC SHEAR IN THE REPRESENTATION OF THE MECHANICAL BEHAVIOR OF LAMINATED COMPOSITES

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Key words: Strain Gradient Notation, Artificial Stiffening, Qualitative Errors, Laminated Composites.

Abstract. *This paper presents a study of the deleterious effects caused by the presence of parasitic shear in the polynomial expansions of finite elements for modeling laminated composite plates and beams. It is shown that spurious terms are responsible not only for artificial stiffening, but also cause qualitative errors in the behavior of laminated composites. That is, parasitic shear is capable of causing the finite element to misrepresent the nature of a given deformation. The sources of parasitic shear are discussed in the context of beams and plates. Through the use of strain gradient notation, the parasitic shear terms are precisely identified, and a sensitivity analysis can reveal the strength of each and every term in the deleterious effects of the group of spurious terms. Correction of the elements is possible by removing these spurious terms. Numerical analysis demonstrates the convergence characteristics of the elements contaminated by parasitic shear and that in certain cases the only definite remedy is the removal of the spurious terms a-priori.*

1 INTRODUCTION

Errors can be introduced into the finite element model at the individual element level which make the element overly stiff. This artificial stiffening is thus a quantitative error, i.e. an error in magnitude. One source of artificial stiffening is known as parasitic shear which is caused by incorrect coupling between flexural and shear deformations. Parasitic shear also introduces qualitative errors into laminated composite models. A qualitative error is defined as a misrepresentation of the nature of a deformation. For instance, it can be an error in the form of the represented solution throughout the element. That is the case in a shear deformable laminated beam where a constant shear force is misrepresented by a parabola[1]. Another example of qualitative error is the misrepresentation of the direction of a certain displacement. This occurs in cantilever laminated composite plates[2].

The source of parasitic shear is the use of incompatible polynomials as approximations for the field variables. Incompatibility of polynomials manifest itself in two different ways. The polynomial may be incomplete such as is the case in the displacement representation of a four-node quadrilateral element for modeling plane problems. Further, as it occurs in plate analysis, the polynomial may be inconsistent. That is, the polynomial is incomplete and its order is inconsistent with the order of the theory being modeled. The spurious terms present into the strains approximations which are associated to parasitic shear may be clearly identified and eliminated if a physically interpretable notation is employed. Strain gradient notation[3] is used here to formulate laminated composite beam and plate elements. This notation was first developed to determine the equivalent continnum properties of lattice structures[4]. Then, its use was extended to analyse the modeling behavior of individual finite elements[5]. Further, strain gradient notation was applied in a procedure for formulating finite elements stiffness matrices. This latter application has provided insight into finite elements formulation as it has allowed for the identification of spurious terms present in the elements´polynomials[6].

In the following sections aspects of the formulation of strain gradient finite elements for laminated composites are highlighted. Parasitic shear terms are identified and a straightforward procedure for eliminating them is outlined. Single-element plate models are analyzed to show the nature of qualitative errors present. Refinement of the models are performed to study the convergence characteristics of these qualitative errors. Results will indicate that certain errors cannot be removed through refinement, and thus removal of the erroneous terms is necessary prior to computational implementation.

2 MACROMECHANICAL THEORY FOR LAMINATED COMPOSITES

This section presents the macromechanical theory adopted for modeling laminated composite components. As the objective of the work is to discuss the detrimental effects of parasitic shear in laminated composite finite elements, the simple theory of the equivalent lamina is employed[7, 8].

The development is based on the following four assumptions: (1) the laminate is comprised of homogeneous, orthotropic laminae, (2) the strain and stress components normal to the middle surface of the laminate are small when compared to the other components and are neglected, (3) there is a perfect bond between the laminae, which prevents the laminae form slipping relatively to each other or deforming independently and (4) the laminate exhibits membrane behavior. The second and third assumptions allow the laminated composite body to be considered as an equivalent single-layer body so that the behavior of the laminated composite is entirely represented by the behavior of its middle surface.

In the next section, expressions for plate and beam finite elements based on strain gradient notation are developed.

3 LAMINATED COMPOSITE FINITE ELEMENTS

Here polynomial expansions for displacements and strains based on strain gradient notation are presented for a four-node plate and a two-node beam for modeling shear deformable laminated composite structural components. The coefficients of strain gradient notation polynomials are written in terms of physical quantities which are the sources of displacements and, consequently, strains in the continnum. These sources are kinematic quantities; namely, rigid body motions, strains and derivatives of strains (or strain gradients).

3.1 Four-Node Plate

The independent field variables in a shear deformable plate are the displacements u and v which are parallel to the middle surface, the out-of-plane displacement w, and the rotations p and q. The displacements u and v are defined as the sum of the middle surface in-plane displacements uo and vo, and the flexure terms linear in z, u´and v´ that are associated with the rotations q and p, respectively. The rotations q and p are defined as the first derivatives of the in-plane displacements u and v with respect to the variable z.

Each node of the finite element has five degrees-of-freedom; namely, displacements u, v and w, and rotations p and q. The polynomial expansions in strain gradient notation for the five field variables are

$$\begin{aligned} w(x,y) = {} & (w_{rb})_o + (\gamma_{xz}/2 - q_{rb})_o x + (\gamma_{yz}/2 - p_{rb})_o y \\ & + \left[\left(-\gamma_{xy,z} + \gamma_{yz,x} + \gamma_{xz,y}\right)\right]_o xy \end{aligned} \tag{1a}$$

$$\begin{aligned} u(x,y,z) = {} & (u_{rb})_o + (\varepsilon_x)_o x + (\gamma_{xy}/2 - r_{rb})_o y + (\varepsilon_{x,y})_o xy \\ & + (\gamma_{xz}/2 - q_{rb})_o z + (\varepsilon_{x,z})_o xz + (\varepsilon_{x,yz})_o xyz \\ & + \left[\left(\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y}\right)/2\right]_o yz \end{aligned} \tag{1b}$$

$$\begin{aligned} v(x,y,z) = {} & (v_{rb})_o + (\gamma_{xy}/2 + r_{rb})_o x + (\varepsilon_y)_o y + (\varepsilon_{y,x})_o xy \\ & + (\gamma_{yz}/2 - p_{rb})_o z + (\varepsilon_{y,z})_o yz + (\varepsilon_{y,xz})_o xyz \\ & + \left[\left(\gamma_{xy,z} + \gamma_{yz,x} - \gamma_{xz,y}\right)/2\right]_o xz \end{aligned} \tag{1c}$$

$$q(x,y) = (\gamma_{xz}/2 - q_{rb})_o + (\varepsilon_{x,z})_o x + (\varepsilon_{x,yz})_o xy \\ + \left[(\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y})\ /2\right]_o y \tag{1d}$$

$$p(x,y) = (p_{rb} - \gamma_{yz}/2)_o - (\varepsilon_{y,z})_o y - (\varepsilon_{y,xz})_o xy \\ - \left[(\gamma_{xy,z} + \gamma_{yz,x} - \gamma_{xz,y})/2\right]_o xz \tag{1e}$$

These displacement approximations contain the 20 independent strain states that the element is capable of representing. They are

Rigid body modes

$$(u_{rb})_o \quad (v_{rb})_o \quad (w_{rb})_o \quad (p_{rb})_o \quad (q_{rb})_o \quad (r_{rb})_o \tag{2a}$$

Constant strains

$$(\varepsilon_{\mathrm{x}})_o \quad (\varepsilon_{\mathrm{y}})_o \quad (\gamma_{\mathrm{xy}})_o \quad (\gamma_{\mathrm{xz}})_o \quad (\gamma_{\mathrm{yz}})_o \tag{2b}$$

Normal gradients

$$(\varepsilon_{\mathrm{x,y}})_o \quad (\varepsilon_{\mathrm{x,z}})_o \quad (\varepsilon_{\mathrm{x,yz}})_o \quad (\varepsilon_{\mathrm{y,x}})_o \quad (\varepsilon_{\mathrm{y,z}})_o \quad (\varepsilon_{\mathrm{y,xz}})_o \tag{2c}$$

Shear gradients

$$(\gamma_{\mathrm{xy,z}})_o \quad (\gamma_{\mathrm{xz,y}})_o \quad (\gamma_{\mathrm{yz,x}})_o \tag{2d}$$

Examination of these strain gradient quantities shows that the element embodies the transverse shear effects as proposed, and also the main convergence requirements for finite elements, i.e. rigid body modes and constant strains.

The elastic strain expansions are obtained from the displacement polynomials according to the definitions of elasticity theory

$$\varepsilon_x = (\varepsilon_x)_o + (\varepsilon_{x,y})_o y + (\varepsilon_{x,z})_o z + (\varepsilon_{x,yz})_o yz \tag{3a}$$

$$\varepsilon_y = (\varepsilon_y)_o + (\varepsilon_{y,x})_o x + (\varepsilon_{y,z})_o z + (\varepsilon_{y,xz})_o xz \tag{3b}$$

$$\gamma_{xy} = (\gamma_{xy})_o + (\gamma_{xy,z})_o z + (\varepsilon_{x,y})_o x + (\varepsilon_{y,x})_o y \\ + (\varepsilon_{x,yz})_o xz + (\varepsilon_{y,xz})_o yz \tag{3c}$$

$$\gamma_{yz} = (\gamma_{yz})_o + (\gamma_{yz,x})_o x + (\varepsilon_{y,z})_o y + (\varepsilon_{y,xz})_o xy \tag{3d}$$

$$\gamma_{xz} = (\gamma_{xz})_o + (\gamma_{xz,z})_o y + (\varepsilon_{x,z})_o x + (\varepsilon_{x,yz})_o xy \tag{3e}$$

These expressions show the physical terms, which contribute to the quantification of the elastic strains. Careful inspection of them reveals the presence of spurious terms in the shear strain expansions. These spurious terms are gradients of normal strains, which do not belong to the Taylor series expansions of shear strains. Physically, normal strains and their gradients do not contribute for quantifying shear strains since they are independent. The erroneous terms present in equations 3c, d and e are parasitic shear, which are responsible for deleterious effects present in laminated composites analysis. In the analysis of homogeneous materials, parasitic shear causes artificial stiffening. That is, finite elements are made overly stiff and convergence is attained only after great deal of refinement. In those cases, the error introduced by parasitic shear is a quantitative error. In laminated composites, however, parasitic shear also causes qualitative errors. Examples of qualitative errors and their convergence characteristics will be shown later in this work.

3.2 Two-Node Beam

The shear deformable beam may be viewed as a two-dimensional version of the four-node plate described above. The beam´s independent variables are the in-plane displacement u, the out-of-plane displacement w and the rotation q. The finite element used to describe the beam is a two-node bar containing three degrees-of-freedom per node; namely, displacements u, w, and rotation q.

The displacement polynomials are written below in strain gradient notation.

$$w(x,y)=(w_{rb})_o+(\gamma_{xz}/2-q_{rb})_o x \tag{4a}$$

$$u(x,z)=(u_{rb})_o+(\varepsilon_x)_o x+(\gamma_{xz}/2-q_{rb})_o z+(\varepsilon_{x,z})_o xz \tag{4b}$$

$$q(x,y)=(\gamma_{xz}/2-q_{rb})_o+(\varepsilon_{x,z})_o x \tag{4c}$$

Analogously, inspection these expressions shows the presence of three rigid body modes, two constant strains and a gradient of normal strain. The elastic strain expressions are given below

$$\varepsilon_x=(\varepsilon_x)_o+(\varepsilon_{x,z})_o z \tag{5a}$$

$$\gamma_{xz}=(\gamma_{xz})_o+(\varepsilon_{x,z})_o x \tag{5b}$$

which shows the presence of a normal strain gradient in the shear strain expansion which is a spurious term responsible for parasitic shear.

The *a-priori* error analysis just shown here and in the previous section has precisely identified the sources of parasitic shear. The elimination of the spurious terms is thus straightforward, i.e. they can be simply removed from the shear strain expansions, rendering corrected elements.

3.3 Stiffness matrix

The strain expansions formed above are substituted into the strain energy expression of a laminated composite plate to produce the element´s stiffness matrix. The strain energy of the laminate is comprised by the sum of the strain energies of all laminae. Since the constitutive relations of a lamina must be transformed from the principal material directions to the laminate´s global directions, coupling of deformation modes are introduced into the mechanics of the laminated composite.

Following the standard procedure and strain gradient notation, the stiffness matrix of the element is written as

$$K = \phi^{-T}\left(\sum_{k=1}^{n}\int_{\Omega_k} T_{sgk}^{T}\overline{Q}_k T_{sgk}\, d\Omega_k\right)\phi^{-1} \tag{6}$$

In this expression, ϕ is the matrix which relates the nodal displacements to the strain gradient quantities of the element. The Tsg_k matrix relates the elastic strains to the set of strain gradients associated to lamina k. This matrix is different for each layer because the strain distribution varies from lamina to lamina. Matrix Q_k contains the constitutive properties of lamina k after coordinate transformation, and n is the number of laminae comprising the laminate composite.

When the element is contaminated with parasitic shear, the spurious terms appear in matrix Tsg_k. Thus, clearly, the parasitic shear terms in the stiffness matrix can be identified. The shear portion of the stiffness matrix of an element containing parasitic shear has larger values.

4 SINGLE-PLATE ELEMENT ANALYSIS

The presence and nature of qualitative errors introduced by parasitic shear into laminated composite finite elements are demonstrated in this section. This is accomplished by running single-plate element analyses with and without parasitic shear and comparing the computed deflections. The element is arranged as a cantilever plate having two nodes fixed and the other two nodes free.

4.1 Example 1: Regular Symmetric Angle-Ply Laminate

A regular symmetric angle-ply laminate made of three layers of equal thickness and with fibers oriented at + 30°, -30° and + 30° is subjected to uniform tension loads. This loading activates coupling between normal forces and the in-plane shearing strain in this laminate.

The single-element analysis test reveals a qualitative error in the representation of the in-plane transverse displacement u of the laminate. When parasitic shear is present, both free nodes displace in the negative direction. The u-displacements of these nodes are -0.419×10^{-5} and -0.107×10^{-4}, indicating that the element deforms to the left of the undeformed configuration. After elimination of parasitic shear, however, the nodes move against each other. The u-displacements are now $0{,}689 \times 10^{-6}$ and -0.563×10^{-5}.

Since the qualitative error just described is an in-plane phenomenon, it is expected to be due to spurious terms associated with in-plane strain gradients. Indeed, a sensitivity analysis reveals that the strain gradients $\varepsilon_{x,y}$ and $\varepsilon_{y,x}$ which are, respectively, the change of the transverse normal strain in the longitudinal direction and the change of the longitudinal normal strain in the transverse direction, are solely responsible for this qualitative error. If only these two strain gradients are removed from the shear strain expressions, the correct behavior of the element is reproduced.

4.2 Example 2: General Non-Symmetric Laminate

A four-layer general non-symmetric laminate is subjected to uniform in-plane shear loads. The thicknesses of the layers are 0.10, 0.08, 0.04 and 0.14 inch, and the orientations of their fibers are, respectively, +30°, +60°, +60° and –20° with respect to the longitudinal edges of the plate. A general non-symmetric laminate possesses all types of coupling of deformation modes.

The qualitative error in this case is related to twisting and bending of the plate. When parasitic shear is present, the free nodes vertical displacements are -0.128×10^{-3} and 0.184×10^{-3}. This shows that the element is twisting with very little bending. On the other hand, after elimination of parasitic shear, the w-displacements of those nodes are 0.399×10^{-2} and 0.382×10^{-2}, which means that now the element is predominantly in bending mode. Therefore, the qualitative error is the inversion of the actual mechanics of the plate element. It is also seen that the displacements are increased by an order of magnitude.

A sensitivity analysis reveals that the strain gradients responsible for the qualitative error described are $\varepsilon_{y,z}$ (flexural deformation) and $\varepsilon_{y,x}$ (in-plane variation of normal strain).

4.3 Example 3: Non-Symmetric Angle-Ply Laminate

A two-layer non-symmetric angle-ply laminate is subjected to a uniform twisting load. The thicknesses of the layers are 0.10 and 0.12 inch, and the orientations of their fibers are, respectively, +30° and –30° with respect to the longitudinal edges of the plate. Due to the non-symmetry with respect to the middle surface, the laminate presents coupling between bending and extension. Further, due to the orientations of the fibers, the laminate possesses coupling between bending and twisting.

When parasitic shear is present, bending is simply suppressed form the results. The nodal out-of-plane displacements are -0.305×10^{-1} and 0.303×10^{-1}, which show that the element is basically twisting. Therefore, the twist-bending coupling is obliterated by parasitic shear. After correction of the element, those displacements become –0.707 and –0.319, indicating that the element bends and twists simultaneously. Further, the displacements are increased by an order of magnitude. A sensitivity analysis has not been performed in this case.

5 CONVERGENCE ANALYSIS

This section demonstrates the convergence characteristics of the qualitative errors described above. The three single-element models are uniformly refined into meshes of 4, 16, 64 and 256 elements, which are referred to as mesh 1, mesh 2, mesh 3 and mesh 4. The results of the models containing parasitic shear are compared to the results of the models corrected for parasitic shear.

5.1 Example 1: Regular Symmetric Angle-Ply Laminate

The single-element model analysis showed that the left edge the plate undergoes an in-plane displacement to the left direction due to parasitic shear. After removal of the spurious terms, this displacement occurs in the opposite direction. The convergence analysis is to show whether the error persists or is removed or either attenuated by mesh refinement. Figure 1(a)-1(d) presents the displacement u along the length of the laminate. The abscissas represent the length coordinate L of the plate, and the ordinates represent the in-plane transverse displacement u. The fixed end of the plate is at L = 30.0 inches.

The plots show that the end of the left edge of the model without parasitic shear displaces to the positive (right) direction, while the end of the left edge of the model with parasitic shear displaces to the negative (left) direction of the undeformed configuration in every refinement. As this opposition in behavior persists through the entire analysis, it can be concluded that the qualitative error is not removed by model refinement in this case. Although there is attenuation of this qualitative error in the representation of the in-plane displacement of the laminate, it is never eliminated, which leads one to assert that the sources of parasitic shear must be removed during element formulation.

Furthermore, the sensitivity analysis results are proven during refinement. The strain gradient terms which cause the present qualitative error are $\varepsilon_{x,y}$ and $\varepsilon_{y,x}$. When only these strain gradients are removed from the shear strain expressions of the element, the correct solutions for this problem are recovered. As depicted in the plots, these solutions match perfectly the solutions obtained when all parasitic shear terms are removed, proving that elimination of the identified parasitic shear terms indeed corrects the plate element.

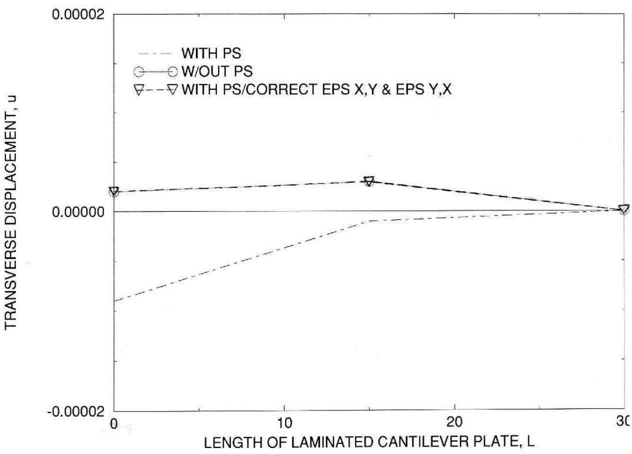

Figure 1(a) - Regular symmetric angle-ply laminate. Mesh 1.

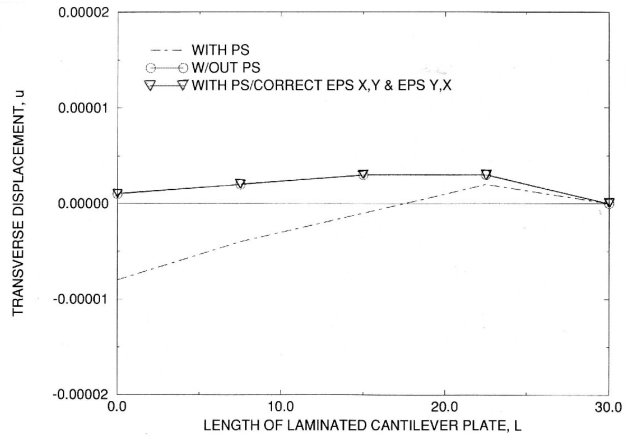

Figure 1(b) - Regular symmetric angle-ply laminate. Mesh 2.

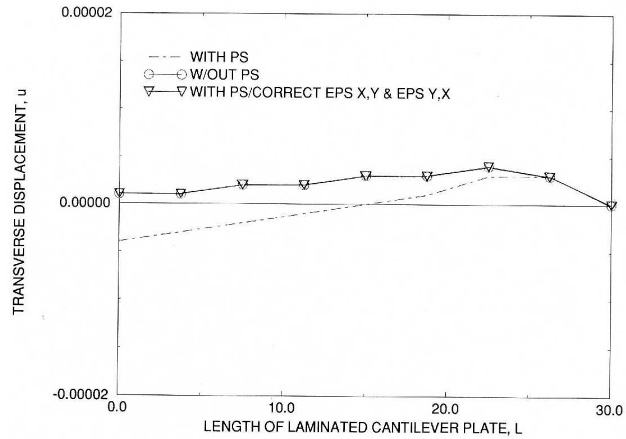

Figure 1(c) - Regular symmetric angle-ply laminate. Mesh 3.

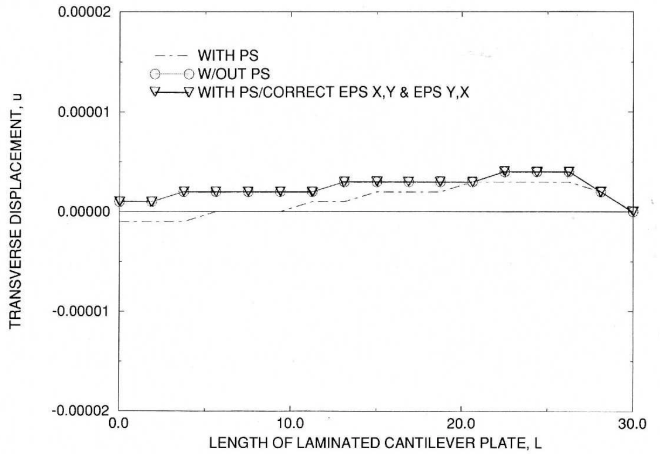

Figure 1(d) - Regular symmetric angle-ply laminate. Mesh 4.

5.2 Example 2: General Non-Symmetric Laminate

The single-element model analysis showed that the coupling between in-plane shear and bending is not represented by the model with parasitic shear. Figures 2(a)-2(c) show that the model without parasitic shear is capable to represent this coupling. On the other hand, the model with parasitic shear is not capable of representing bending when coarse meshes are employed. But, with refinement, as depicted by figure 4(c), the deleterious effect of parasitic shear is attenuated. This is concluded by observing the similarity of the shapes and magnitudes of the solutions provided by the two models in that figure.

Furthermore, convergence analysis verifies that the strain gradients $\varepsilon_{y,z}$ and $\varepsilon_{y,x}$ cause the present qualitative error. The plots show that the solutions provided by the model corrected only for these spurious terms agree qualitatively with the solutions provided by the model entirely corrected for parasitic shear.

5.3 Example 3: Non-Symmetric Angle-Ply Laminate

The single-element model analysis showed that the coupling between twist and bending is not represented by the contaminated model. Figure 3(a)-3(c) depicts the variation of the out-of-plane displacement w of the tip of the plate across its width. They show that the twist-bending coupling is represented by the corrected model. Twisting is observed through rotation of the tip section while bending is manifested through the tip´s downward displacement. It can also be seen that the model with parasitic shear does not represent this coupling regardless of the level of refinement. Therefore, this is another case in which qualitative errors are not even attenuated by mesh refinement.

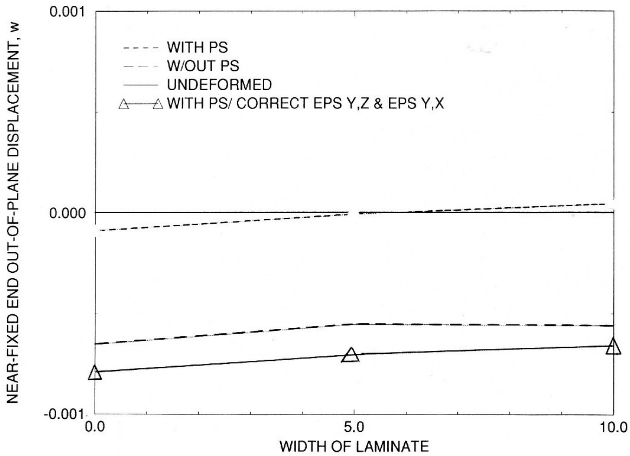

Figure 2(a) – General non-symmetric laminate. Mesh 1.

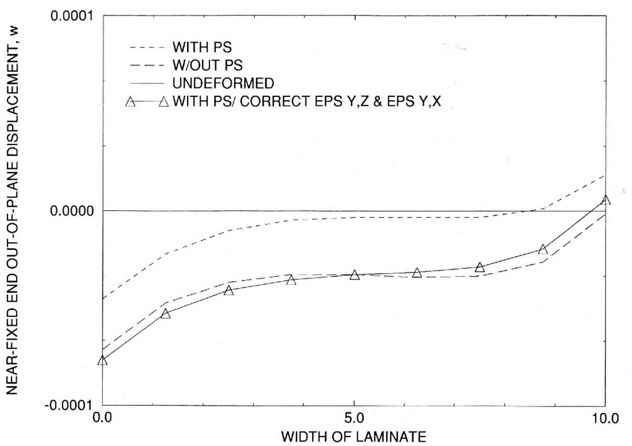

Figure 2(b) – General non-symmetric laminate. Mesh 3.

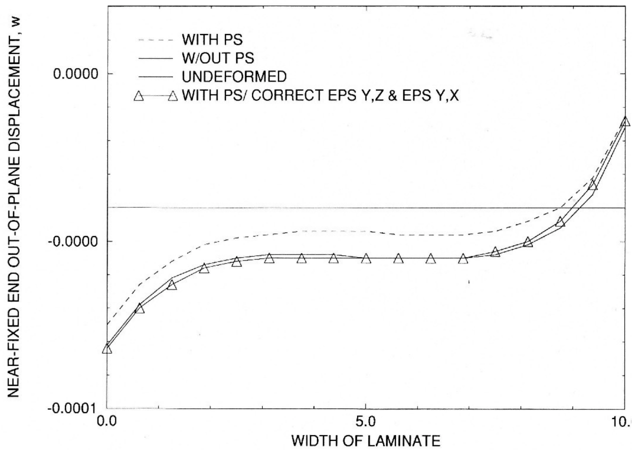

Figure 2(c) – General non-symmetric laminate. Mesh 4.

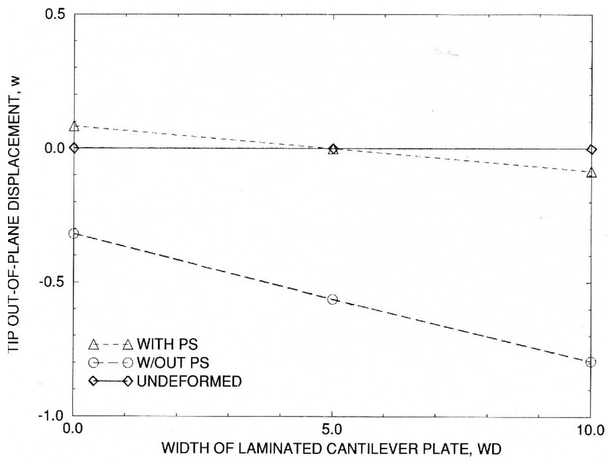

Figure 3(a) – General angle-ply laminate. Mesh 1.

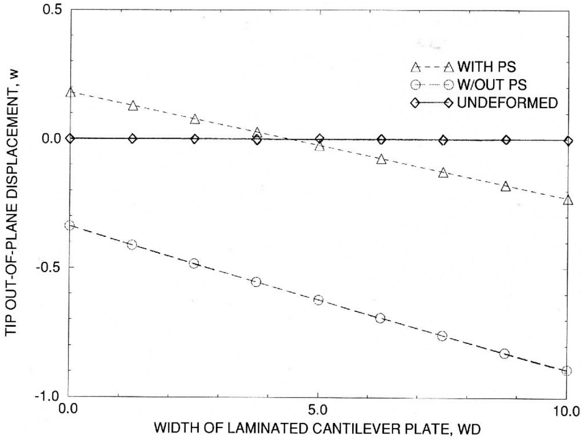

Figure 3(b) – General angle-ply laminate. Mesh 3.

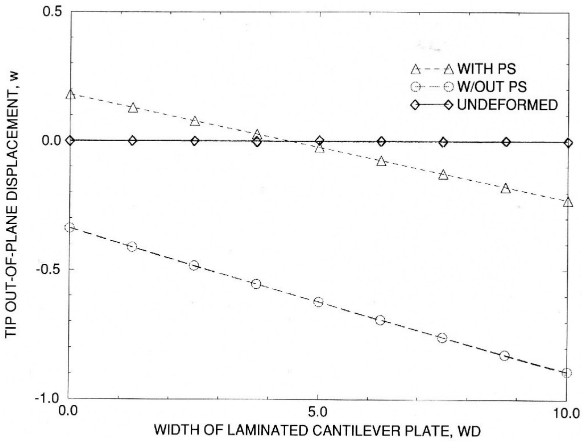

Figure 3(c) – General angle-ply laminate. Mesh 4.

6 ANALYSIS OF A BEAM PROBLEM

Sections 4 and 5 were dedicated to describe the behavior and convergence characteristics of the laminated plate element described previously. Several instances of qualitative errors due to parasitic shear were demonstrated in the plate problems analyzed.

This section is devoted to showing a very serious qualitative error caused by parasitic shear in the beam model. This qualitative error occurs in the representation of the transverse shear force along the beam in the following way. As a tip load is applied, the shear force displays a parabolic variation along the beam when it is actually a constant value. Analysis of the problem with the correct model, produces the expected correct result. Figure 4 shows the values of the transverse shear force along the beam when it is modeled with and without parasitic shear. The plot shows that the erroneous behavior occurs for a refined model leading to the conclusion that the effects of parasitic shear cannot be removed with refinement for this case. Therefore, the spurious term shown in equation 5b has to be removed during element formulation.

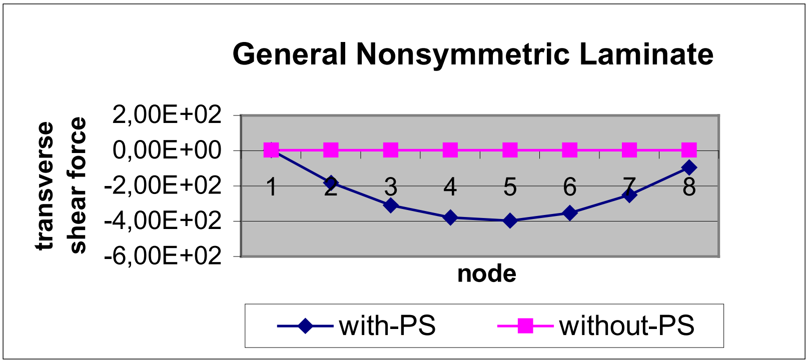

Figure 4 – Transverse shear force on a general non-symmetric laminated beam.

7 CONCLUSION

This work has demonstrated the presence of qualitative errors in laminated composite plate and beam elements, which are caused by parasitic shear. This was done employing strain gradient notation, which is a physically interpretable notation for formulating finite elements. Numerical analyses performed have shown that, at best, qualitative errors may be highly resistant to elimination or either attenuation through refinement. Further, the results have demonstrated that, at worst, qualitative errors cannot be removed by mesh refinement. Two cases in which qualitative errors persist through refinement in plate analysis and one case in beam analysis were shown, thus indicating that the sources of error must be eliminated at the element level if correct analyses are to be performed.

There are different procedures for elimination of spurious terms in the literature, the most widely known being integration using reduced order Gauss quadrature[9]. Although this procedure has been widely and successfully used, the literature also shows that the procedure is not always very effective. In certain cases, for instance, underintegration introduces zero energy modes while eliminating parasitic shear terms[10]. The present work shows that strain gradient notation allows for the correct elimination of the spurious terms because it can identify them precisely.

8 REFERENCES

[1] J.E. Abdalla and R.D. Machado, "Formulation aspects of a laminated composite Timoshenko beam", XX CILAMCE, São Paulo (1999).
[2] J.E. Abdalla, "Qualitative and discretization error analysis of laminated composite plate models", Ph.D. Thesis, University of Colorado, Boulder, CO (1992).
[3] D.E Byrd, "Identification and elimination of errors in finite element analysis", Ph.D. Thesis, University of Colorado, Boulder, CO (1988).
[4] J.O. Dow, Z.W. Su, C.C. Feng and C.S. Bodley, "An equivalent continuum representation of structures composed of repeated elements", AIAA Journal, **23**, 1564-1569 (1985).
[5] J.O. Dow, T.H. Ho, and H.D. Cabiness, "A generalized finite element evaluation procedure", Journal of Structural Engineering, ASCE, Vol. 111, No.ST2, pp 435-452 (1985).
[6] J.O. Dow and D.E Byrd, "The identification and elimination of artificial stiffening errors in finite elements", *Int. J. Num. Meth. Engng*., **26**, 743-762 (1988).
[7] R..M. Jones, *Mechanics of composite materials*, Hemisphere Publishing CO., New York, USA (1975).
[8] J.R. Vinson and R.L. Sierakowski, *The behavior of structures composed of composite materials*, Martinus Nijhoff Publishers, Boston, USA (1986).
[9] O.C. Zienkiewicz and R.L. Taylor, *The finite element method*, McGraw-Hill, Vol. I (1989).
[10] J.O. Dow and J.E. Abdalla, "Qualitative errors in laminated composite plate models", *Int. J. Num. Meth. Engng*., **37**, 1215-1230 (1994).