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THE DYNAMIC STATIONARY BEHAVIOR OF THIN ELASTIC PLATES BY THE BOUNDARY ELEMENT METHOD

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Abstract. *In the present article the stationary dynamic behavior of homogeneous, isotropic and linear elastic Kirchhoff plates is modeled by the Boundary Element Method (BEM). The dynamic stationary fundamental solution of the bi-harmonic equation is used to transform the governing partial differential equations into Boundary Integral Equations (BIE). The BIE is discretized by continuous and/or discontinuous linear elements. After the boundary quantities are determined, domain variables may be easily obtained by an integration procedure. Two displacement integral equations are written for every boundary node. The collocation points of the integral equations are place outside the plate domain, leading to a non-singular BE formulation. In this article the Frequency Response Functions of the thin plates are determined. Modal data, i.e., natural frequencies and the corresponding mode shapes, are obtained from information contained in the FRF. The procedure is validated by comparison with analytical and numerical results available in the literature.*

1 INTRODUCTION

It is well known that the Boundary Element Method (BEM) is an accurate and efficient numerical method for plate dynamic analysis[1]. The transformation of the plate differential equation into an integral equations requires an auxiliary state. There are basically two approaches for treating plate dynamic stationary problems by the BEM[2], according to the type of auxiliary state that is applied. The first method possible strategy makes use of the so-called static fundamental solution. In this case the resulting integral equation presents boundary integrals but also a domain integral. So it is necessary to develop a procedure to deal with the domain integral[3,4]. The second method uses the so called dynamic fundamental solution and the resulting integral equation requires only the discretization of the boundary of the plate being analyzed. The BE formulation can also be divided in direct and indirect, depending on the type of varibles used to set up the problem[5]. If physical quantities are used, the direct formulation is obtained. If ficticious auxiliary varibles are considered, the formulation is called indirect.

The free vibration of plates using the direct version of the BEM was first reported by Vivoli[6] and Vivoli and Filippi[7]. Wong and Hutchinson[8] presented a formulation of the free plate vibration problem by the direct BEM including the effect of corners. Forced vibrations of plates were first considered by Bézine and Gamby[9] by a time domain direct BEM employing constant elements.

In this article, the direct BEM based on the dynamic stationary fundamental solution is presented for the forced and free vibration analysis of elastic plates subjected to time harmonic loadings. The dynamic stationary fundamental solution of the is expressed in terms of the modified Bessel functions. Frequency is explicitly included in the fundamental solution.

In the present BE implementation the geometry of the plate and the variables are discretized using linear elements. At plate corners, where the variables may present discontinuities, discontinuous elements are employed. Corner effects are not taken into account. In thin plate bending problems, every boundary node of a well posed problem presents two unkowns and, consequently, two integral equations are required for the node. One possible choice is to use as the first integral equation, a displacement equation and for the second one, an equation describing the rotation normal to the boundary. The strategy followed in the present article is to use two displacement integral equations for every node. Two distinct collocation points, placed outside the plate domain, are chosen for every node. This leads to a non-sigular BE formulation. The advantage of the formulation is to avoid the need to treat singular integrals. The greatest disadvantage is that it introduces more degrees of freedom related to the coordinates where the collocation points should be best placed.

In this stationary formulation the plate modal quantities, that is, the natural frequencies and the vibration modes, are obtained with the aid of Frequency Response Functions (FRF). A point of the plate is excited by an harmonic force of constant amplitude and the resulting displacement is measured (calculated) at another point. The displacement response for a given range of frequencies constitutes the FRF. The plate operational eigenfrequencies (operational natural frequencies) are obtained from the resonances present in the FRF.

The operational eigenmodes (vibration mode shapes) are determined by calculating the plate displacement field at the determined eigenfrequencies. Interior plate points are used to determined the operational mode shape. The present formulation is applied, exemplarily, to the dynamic analysis of a plate with known analytical solution. It is shown that the implemented procedure delivers accurate operational eigenfrequencies and eigenmodes.

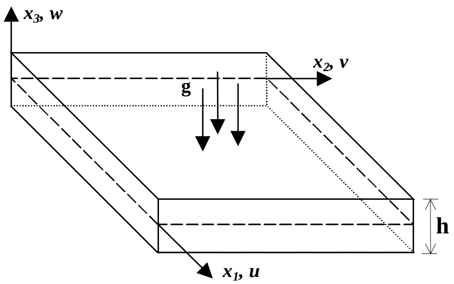

Fig. 1 Coordinate system.

2 BASIC EQUATIONS

A Cartesian coordinate system x_j, (j = 1, 2, 3), shown in figure 1, is used to describe the plate domain. The quantity h is the plate thickness, $-h/2 \leq x_3 \leq h/2$. The displacements are given by u, v, w, respectively. Normal and shear stresses components acting on the plate are designated by σ_{11}, σ_{22}, σ_{33} and τ_{12}, τ_{13}, τ_{23}. The equilibrium equations for an infinitesimal plate element under a dynamical transverse loading $g = g\ (x_1,\ x_2,\ t)$ and in absence of a body force are given by:

$$q_{i,i} + g = \rho\, h\, \ddot{w} \tag{1}$$

$$m_{ij,j} - q_i = 0 \tag{2}$$

where, ρh is the mass density per unit area, overdots indicate differentiation with respect to time t and

$$m_{ij} = \int_{-h/2}^{h/2} \sigma_{ij}\, x_3\, dx_3 \tag{3}$$

are the bending and twisting moment and

$$q_i = \int_{-h/2}^{h/2} \tau_{i3}\, dx_3 \tag{4}$$

are the shear components acting on the element. In thin plate theory, the constitutive relations in terms of out-of-plane displacement w may be written as:

$$m_{ij} = -D\left[(1-\nu)w,_{ij} + \nu\delta_{ij}w,_{kk}\right] \tag{5}$$

$$q_i = -Dw,_{ijj} \tag{6}$$

In equations (5) and (6) $D = Eh^3/12(1-\nu^2)$ is the plate flexural rigidity. Proper substitution of expressions (5) and (6) into equations (2) and (1) yields the standard dynamic equation for thin plates:

$$\nabla^4 w + \frac{\rho h}{D}\ddot{w} = \frac{g}{D} \tag{7}$$

Consider that all variables are undegoing a time harmonic displacement, u(t)= û exp(iωt)with circular frequency ω. Under this circunstance, load g and deflections w will also vary harmonically:

$$g(x_1, x_2, t) = \hat{g}(x_1, x_2)\exp[i\omega t] \tag{8a}$$

$$w(x_1, x_2, t) = \hat{w}(x_1, x_2)\exp[i\omega t] \tag{8b}$$

Substituting relations (8) into equation (7) yields the equation governing the dynamic stationary behavior of Kirchhoff plates:

$$\nabla^4 \hat{w} - \eta^4 \hat{w} = \frac{\hat{g}}{D} \tag{9}$$

where

$$\eta^4 = \frac{\rho h \omega^2}{D} \tag{10}$$

Considering the force and moment conditions at the plate boundary Γ, the twisting moment M_{ns} is usually integrated with the shear force q_n acting on the boundary, resulting in the equivalent shear force $\mathbf{V}_n$, which is given by:

$$V_n = q_n + M_{ns,s} \tag{11}$$

Boundary conditions are expressed in terms of the displacement w, the normal slope $w,_n$, the bending moment M_n and equivalent shear force V_n where n and s represent the outward normal and the tangential coordinates at the boundary Γ, respectively (Fig. 2). Usual boundary conditions are:

Clamped: $\quad w = 0, \quad w,_n = 0 \quad$ (12)

Simply Supported: $\quad w = 0, \quad M_n = 0 \quad$ (13)

Free: $\quad V_n = 0, \quad M_n = 0 \quad$ (14)

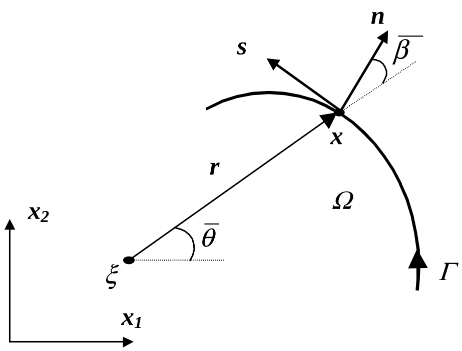

Fig. 2 Normal n and tangential s coordinates at a boundary point ***x***.

3 FUNDAMENTAL SOLUTION

The displacement fundamental solution of equation (9) is given by $\boldsymbol{w}^* = \boldsymbol{w}^*(\xi, \boldsymbol{x})$. It is the solution of equation (9) with a Dirac's Delta distribution as the load g applied at the poin ξ:

$$\nabla^4 \boldsymbol{w}^* - \eta^4 \boldsymbol{w}^* = -\delta(\xi, \boldsymbol{x}) \tag{15}$$

It represents the transversal deflection of an infinitely extended plate at point ***x*** due to a unit concentrated lateral load at point ξ (Fig. 2). The solution of (15) has the form[7, 10, 11]

$$\boldsymbol{w}^* = -\boldsymbol{i}\,\boldsymbol{C}_1 \boldsymbol{J}_0(\eta \boldsymbol{r}) + \boldsymbol{C}_1 \boldsymbol{Y}_0(\eta \boldsymbol{r}) + \boldsymbol{C}_2 \boldsymbol{K}_0(\eta \boldsymbol{r}) \tag{16}$$

with

$$\boldsymbol{C}_1 = \frac{\boldsymbol{1}}{\boldsymbol{8}\eta^2}, \qquad \boldsymbol{C}_2 = \frac{\boldsymbol{1}}{\boldsymbol{4}\pi\eta^2} \tag{17}$$

In equation (16) $\boldsymbol{J_0}$ e $\boldsymbol{Y_0}$ are the zero order Bessel functions of the first and second kind, respectively, $\boldsymbol{K_0}$ is the zero order modified Bessel function of the second kind, $\boldsymbol{i} = \sqrt{-1}$, and $\boldsymbol{r} = |\xi - \boldsymbol{x}|$. Explicit expressions for remaining fundamentals solutions, in terms of $\overset{\circ}{w}_{,n}$, M_n^* and V_n^* are as follows [4, 7, 8]:

$$\boldsymbol{w}_{,n}^* = \boldsymbol{i}\,\boldsymbol{C}_1 \eta \boldsymbol{J}_1(\eta \boldsymbol{r}) \boldsymbol{\cos}\,\overline{\beta} - \eta\left[\boldsymbol{C}_1 \boldsymbol{Y}_1(\eta \boldsymbol{r}) + \boldsymbol{C}_2 \boldsymbol{K}_1(\eta \boldsymbol{r})\right] \boldsymbol{\cos}\,\overline{\beta} \tag{18}$$

$$M_n^* = -i\left\{C_1 \frac{D}{2}\left[1+\nu+(1-\nu)\cos 2\overline{\beta}\right]\eta^2 J_0(\eta r) - C_1 D\eta(1-\nu)\frac{J_1(\eta r)}{r}\cos 2\overline{\beta}\right\}$$
$$+\frac{D}{2}\left\{\eta^2\left[1+\nu+(1-\nu)\cos 2\overline{\beta}\right]\left[C_1 Y_0(\eta r) - C_2 K_0(\eta r)\right]\right.$$
$$\left. -2\eta(1-\nu)\frac{1}{r}\left[C_1 Y_1(\eta r) + C_2 K_1(\eta r)\right]\cos 2\overline{\beta}\right\} \tag{19}$$

$$V_n^* = iC_1 D\left\{J_1(\eta r)\left[\eta^3\cos\overline{\beta} + \frac{\eta^3(1-\nu)}{2} sen\, 2\overline{\beta}\ sen\,\overline{\beta} + \frac{2\eta(1-\nu)}{r}\left(\frac{\cos 3\overline{\beta}}{r} - \frac{\cos 2\overline{\beta}}{R}\right)\right]\right.$$
$$\left. +(1-\nu)\eta^2 J_0(\eta r)\left(\frac{\cos 2\overline{\beta}}{R} - \frac{\cos 3\overline{\beta}}{r}\right)\right\} - D\eta^3\left[C_1 Y_1(\eta r) - C_2 K_1(\eta r)\right]\cos\overline{\beta}$$
$$+D(1-\nu)\left\{\frac{\eta^2}{r}\left[C_1 Y_0(\eta r) - C_2 K_0(\eta r)\right] - \frac{2\eta}{r^2}\left[C_1 Y_1(\eta r) + C_2 K_1(\eta r)\right]\cos 3\overline{\beta}\right\}$$
$$-D(1-\nu)\left\{\frac{\eta^2}{R}\left[C_1 Y_0(\eta r) - C_2 K_0(\eta r)\right] - \frac{2\eta}{rR}\left[C_1 Y_1(\eta r) + C_2 K_1(\eta r)\right]\cos 2\overline{\beta}\right\}$$
$$-\frac{D(1-\nu)}{2}\eta^3\left[C_1 Y_1(\eta r) - C_2 K_1(\eta r)\right] sen\, 2\overline{\beta}\ sen\,\overline{\beta} \tag{20}$$

4 BOUNDARY INTEGRAL FORMULATION

The next step in the Boundary Element Method formulation is to transform the differential equation (9) into a boundary integral equation[2]. With the aid of a reciprocal work theorem and the fundamental solution (16) and its derivatives (18) to (20), it is possible to write an integral equation for the displacement components $\boldsymbol{w}$ at a point ξ, $\boldsymbol{w}(\xi)$:

$$C_P w(\xi) + \int_\Gamma \left[V_n^*(\xi,x)w(x) - M_n^*(\xi,x)w_{,n}(x)\right]d\Gamma(x) + \sum_{i=1}^{N_c} R_{ci}^*(\xi,x_c)w_{ci}(x_c) =$$
$$\int_\Gamma \left[V_n(x)w^*(\xi,x) - M_n(x)w_{,n}^*(\xi,x)\right]d\Gamma(x) + \sum_{i=1}^{N_c} R_{ci}(x_c)w_{ci}^*(\xi,x_c) +$$
$$+\int_\Omega g(X)\,w^*(\xi,X)\,d\Omega(X) \tag{21}$$

with,

$$R_{ci} = \left(M_{ns}^F - M_{ns}^B\right)_i \tag{22}$$

In equation (21) $\boldsymbol{C_P}$ depends only upon the geometry of the boundary and it equals to *1*/2 for the case of a smooth boundary, $\boldsymbol{N_c}$ is the number of corners; $\boldsymbol{R_{ci}}$ is the corner reaction that is related to twisting moment $\boldsymbol{M_{ns}}$ in the forward (***F***) and backward (***B***) neighborhood of the i-th corner; $\boldsymbol{w,_n^*}$, $\boldsymbol{M_n^*}$, $\boldsymbol{V_n^*}$, $\boldsymbol{R_{ci}^*}$ and $\boldsymbol{w_{ci}^*}$ represent the elements derived from the fundamentals solution $\boldsymbol{w^*}$ and, finally, $\boldsymbol{g}(\boldsymbol{X})$ is the load applied perpendicularly to the plate surface.

The boundary equation (21) can be discretized by dividing the boundary Γ into N_e elements. Within every element Γ_e the generalized displacements and forces may be approximated by a polynomial interpolation. In the present case linear interpolation functions (shape functions) ϕ_m are used and the resulting equation may be written in terms of the nodal variables w_k^m, $w_{,nk}^m$, $\boldsymbol{w}_{ci}$, V_{nk}^m, M_{nk}^m, $\boldsymbol{R}_{ci}$:

$$
\begin{aligned}
&C_p w(\xi) + \sum_{j=1}^{N_e} w_k^m \int_{\Gamma_j} V_n^*(\xi, x)\phi_m(x) d\Gamma_j(x) - \sum_{j=1}^{N_e} w,_{nk}^m \int_{\Gamma_j} M_n^*(\xi, x)\phi_m(x) d\Gamma_j(x) + \\
&+ \sum_{i=1}^{N_c} R_{ci}^*(\xi, x_c) w_{ci}(x_c) = \sum_{j=1}^{N_e} V_{nk}^m \int_{\Gamma_j} w^*(\xi, x)\phi_m(x) d\Gamma_j(x) - \sum_{j=1}^{N_e} M_{nk}^m \int_{\Gamma_j} w,_n^*(\xi, x)\phi_m(x) d\Gamma_j(x) + \\
&+ \sum_{i=1}^{N_c} R_{ci}(x_c) w_{ci}^*(\xi, x_c) + \int_{\Omega} g(X)\, w^*(\xi, X)\, d\Omega(X) \qquad (23)
\end{aligned}
$$

In the present implementation the collocation point ξ is placed outside the plate domain ($\xi \notin \Omega$) and the integration free-term disappears, C_p=0. Moreover, the corner reactions $\boldsymbol{R}_{ci}$ can be written in terms of neighbor node rotations using a finite difference scheme. Although this is the correct way to treat corner reactions, in the present implementation these terms were neglected. The accurate obtained results obtained neglecting these terms showed it to be a reasonable approximation.

5 FORMULATION AND SOLUTION OF THE LINEAR EQUATION SYSTEM

The plate boundary described in equation (23) was discretized by rectilinear elements described by linear shape functions. Considering B_1 and B_2 the initial and final coordintates of the elements, the element geometry may be expressed in terms of intrinsic coordinates, ς:

$$
b(\varsigma) = B_1 \frac{1-\varsigma}{2} + B_2 \frac{1+\varsigma}{2} \qquad (24)
$$

This same interpolation is used for the varibles of the elements possessing no corners, leading to an isoparametrical formulation. For elements with corners the field variables were discretized by discontinuous elements, as shown in figure 3. The corner nodes were displaced towards the interior by one fourth of the element length, L_e/4.

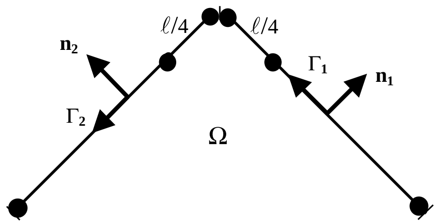

Figure 3: Discontinuous linear element.

Two integral equations were writtten for every boundary node. The collocation points were placed outside the plate domains, at distances d_1 and d_2, respectively, as shown in figure 4, leading to non-singular integral equations.

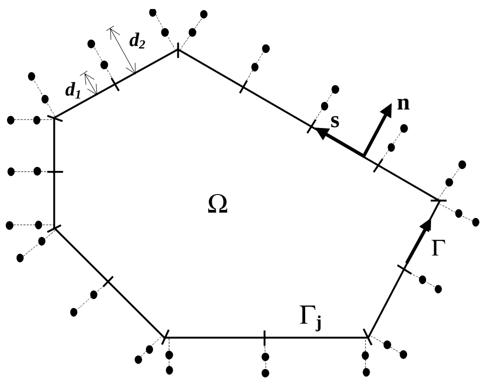

Figure 4: Collocation points for non-sigular BE plate implementation

The coefficients of the system matrices H and G were determined integrating the product of fundamental solution kernels, shape functions and Jacobians over the elements. These non-singular integrations were performed with standard Gaussian quadrature. This procedure leads to a linear system of algebraic equations:

$$[\boldsymbol{H}]\{\boldsymbol{U}\} = [\boldsymbol{G}]\{\boldsymbol{T}\} \tag{25}$$

where ***U*** and ***T*** contain generalized boundary displacement and forces vectors, respectively. After the plate boundary conditions have been introduced in equation (25) the resulting system may be written:

$$[\boldsymbol{A}]\{\boldsymbol{X}\}=\{\boldsymbol{B}\} \tag{26}$$

The solution of equation (26), the vector X, contains all unknown boundary quantities. The system matrix [A(ω)] contain frequency dependent terms. After the vector X is determined, the displacement at the plate domain may be readily obtained by the non singular integrations indicated in equation (21).

6 NUMERICAL EXAMPLE

In this section the formulation described previously will be validated by means of a simple plate example, shown in figure 5. The plate is square, two edges are clamped and two are free. The plate is excited by an unit harmonic normal force F, applied at one of the boundary nodes (node 20). The remaining plate data are:

Poisson's ratio	$\nu = 0.3$
Young's modulus	$\boldsymbol{E}$ = 1000 N/m^2
Thickness	$\boldsymbol{h}$ = 0.05 m
Density	ρ = 0.229 Kg/m^3
Plate length	l=1.0 m
Plate width	b=1.0m

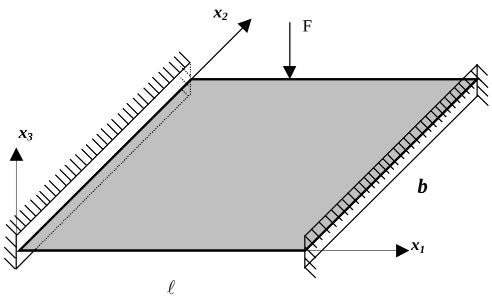

Fig. 5 Square plate subjected to concentrated time-harmonic load.

Computations are carried out for the following two boundary discretizations with using linear elements:

i) 40 boundary elements, 4 double nodes (discontinuous elements), 44 boundary nodes
ii) 80 boundary elements, 4 double nodes (discontinuous elements), 84 boundary nodes

The first discretization of boundary and domain is shown in Fig. 6. To obtain the operational eingenmodes, the displacement at 49 interior points were determined (see fig.6). The distance from de colocation points to the boundary elements are assumed to be $d_1=0.25L_e$ and $d_2=0.50L_e$ (see fig4). In these relations L_e is the length of the boundary element near the collocation point. Numerical studies indicated that these positions for the collocation points lead to accurate results.

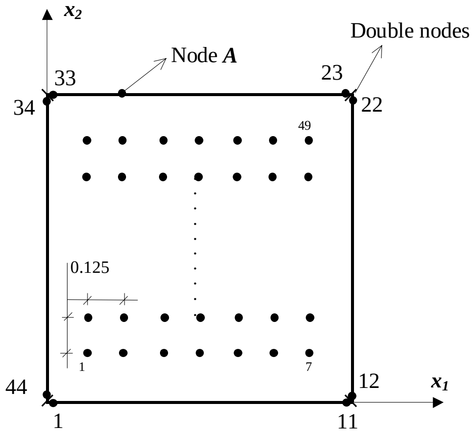

Figure 6: Boundary and domain discretization for the square plate.

The analytical solution for the eigenfrequencies of the square plate is given by the following expression[12]:

$$\omega_{ij} = \frac{\lambda_{ij}^2}{2\pi\ell^2}\left[\frac{Eh^3}{12\gamma\left(1-\nu^2\right)}\right]^{\frac{1}{2}} \tag{27}$$

where, w_{ij} are the natural frequencies, ℓ is the length of plate, b is the plate width, h is the plate thickness, i is the number of half-waves in mode shape along horizontal axis x_1, j is the number of half-waves in mode shape along vertical axis x_2, γ is the mass per unit area of plate (ρh for a plate of a material with density ρ), ν is the Poisson ratio and λ_{ij} is a dimensionless frequency parameter.

The value of the dimensionless frequency parameter λ_{ij} for the first six eigenfrequencies of the square plate are reproduced in Table 1. The indices (ij) for every eigenfrequency are also shown in table 1. If the values of the frequency parameters λ_{ij} are known, the analytical value of the plate eigenfrequencies be determined using equation (27).

Table 1 Frequency parameter (λ_{ij}^2) of the square plate.

	λ_{ij}^2 (*i j*)					
	Eigenvalue sequence					
ℓ/**b**	**1**	**2**	**3**	**4**	**5**	**6**
1.0	22.27 (11)	26.53 (12)	43.66 (13)	61.47 (21)	67.55 (22)	79.90 (14)

The plate $FRFs_{20\text{-}20}$, that means, the frequency response functions obtained by exciting the node number 20 and calculating the response at the same point (node 20) are shown in figures 7 and 8, for the two discretizations mentioned above. In these two figures the resonances and anti-resonances can be clearly recognized. The system operational eigenfrequencies (natural frequencies) are determined from the frequencies at which resonances in the FRFs occur. Table 2 reports the value of the initial six eigenfrequencies obtained by this procedure and also the value of the analytical solution. It can be seen that both discretization cases produce accurate results for the plate natural frequencies.

The other modal quantity necessary to characterize the stationary dynamic behavior of the plate is given by the eigenmodes or the natural modes of vibration. In the present case the operational modes are obtained by calculating the displacement field within the plate at each resonance frequency present in the FRF. Figures 9 to 15 show the first six eigenmodes as determined from calculating the plate displacement.$w(x_1, x_2)$ at the 49 internal nodes. The reported operational modes agree with the theoretical eigenmodes of the analytical solution[12,13]. It should be stressed that only the internal displacements are plotted to characterize the operational eigenmodes. Boundary data can be easily incorporated in these figures.

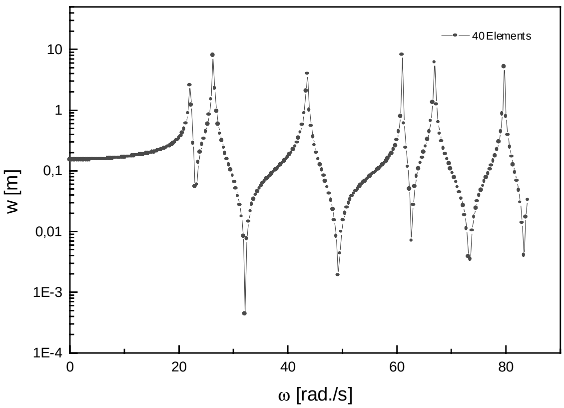

Figure 7: $FRF_{20\text{-}20}$ for the first plate discretization, 40 linear elements.

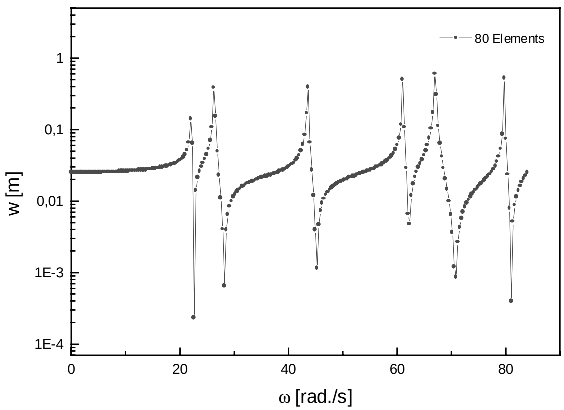

Figure 8: $FRF_{20\text{-}20}$ for the first plate discretization, 80 linear elements.

Table 2 First six analytical and numerical frequencies (w_{ij}) of the square plate.

Discretization	ω_{ij} e (*i j*)					
	1	**2**	**3**	**4**	**5**	**6**
40 elements	22.156	26.374	43.599	61.180	66.799	79.806
80 elements	22.156	26.374	43.599	61.180	67.151	79.810
Analytical	**22.2670 (11)**	**26.5264 (12)**	**43.6541 (13)**	**61.4616 (21)**	**67.5408 (22)**	**79.8891 (14)**

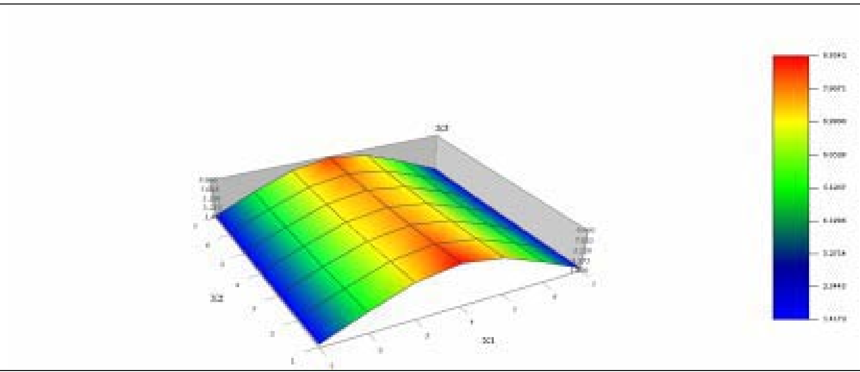

Figure 9: First operational eigenmode of the square plate (ω_1=22.15rad/s).

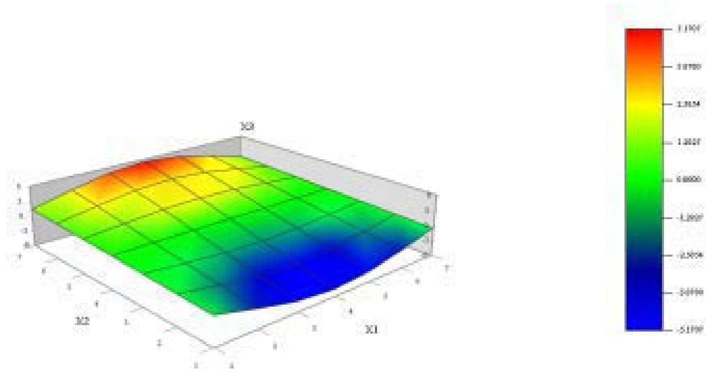

Figure 10: Second operational eigenmode of the square plate (ω_2=26.37rad/s).

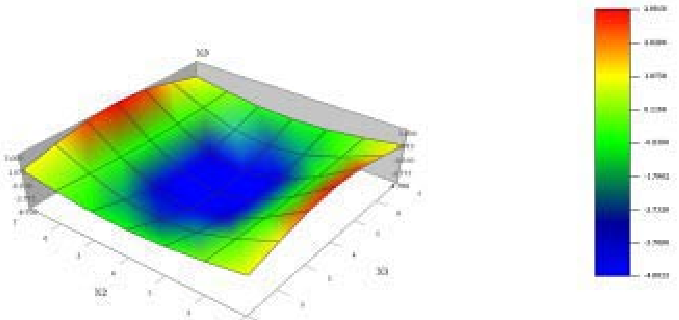

Figure 12: Third operational eigenmode of the square plate (ω_3=43.59rad/s).

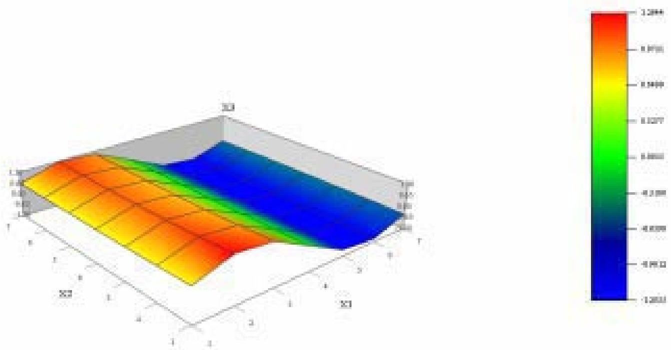

Figure 13: Fourth operational eigenmode of the square plate (ω_4=61.18rad/s).

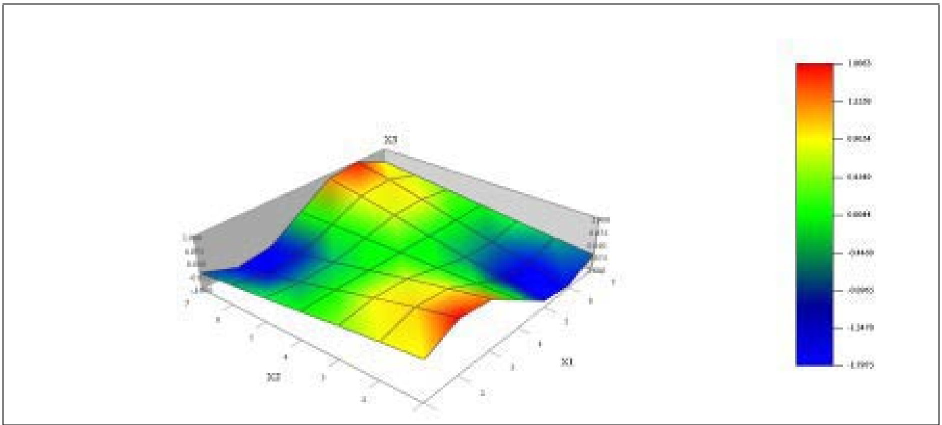

Figure 14: Fifth operational eigenmode of the square plate (ω_5=66.79rad/s).

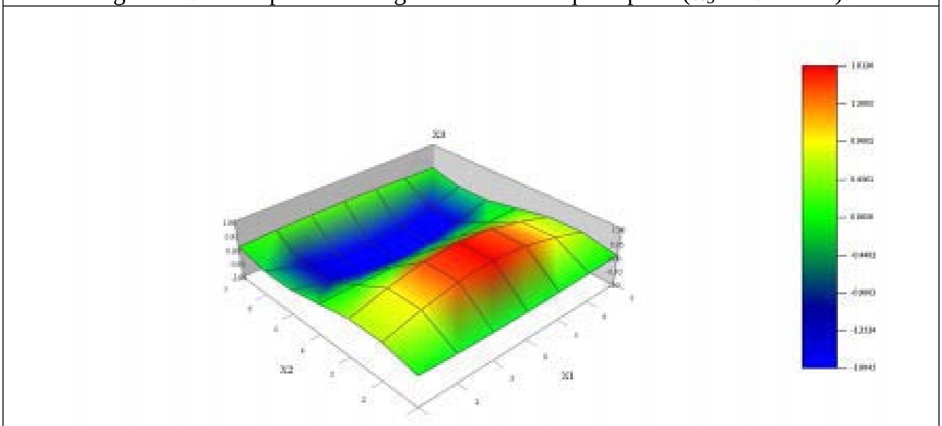

Figure 15: Sixth operational eigenmode of the square plate (ω_6=79.80rad/s).

7 CONCLUDING REMARKS

An implementation of the direct version of the Boundary Element Method (BEM) is presented to analyze the stationary dynamic behavior of Kirchhoff plates. The stationary fundamental solution of thin plates is used to transform the differential equation governing the thin plate behavior into a boundary-only integral equation. The boundary equation is discretized using linear continuous and discontinuous linear elements. Two displacement integral equations are written for every boundary node. The collocation points of the integral equations are placed outside de plate domain, leading to a non-singular Boundary Element formulation.

The proposed scheme is used, exemplarily, to obtain modal data, that is, eigenfrequencies and operational eigenmodes of a square plate with two clamped and two free sides. Frequency Response Functions may be determined for every boundary or domain point of the plate. In the reported example, the FRF of a node on a free boundary is used to recover eigenfrequencies. The eigenfrequencies are determined from the resonances of the FRF. At this resonance frequencies the displacement field of the plate furnish the operational eigenmodes. The presented results agree very well with a known analytical solution. The proposed scheme may be seen as an accurate methodology to analyze free and forced stationary vibrations of thin plates, which only requires the discretization of the plate boundary.

8 ACKNOWLEDGEMENTS

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