SMOOTHNESS IMPROVEMENT FOR TEMPERATURES IN TERMS OF THE BESOV REGULARITY OF INITIAL AND DIRICHLET DATA

HUGO AIMAR AND IVANA GÓMEZ

ABSTRACT. Jerison and Kenig in J. Funct. Anal. 130 (1995), no. 1, 161–219, gave a precise region $\mathcal{R}$ in the square $[0,1]^2$ for the pairs $(s, \frac{1}{p})$ for which every harmonic function in the Lipschitz domain D, with Dirichlet data in $B_p^s(\partial D)$, belongs to $B_p^{s+\frac{1}{p}}(D)$. We prove that every temperature u in $\Omega = D \times (0,T)$ belongs to $\mathbb{B}_\tau^\alpha(\Omega)$ with $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$, $0 < \alpha < \min\{d\frac{p-1}{p}, (s+\frac{1}{p})\frac{d}{d-1}\}$ provided that the Dirichlet data f belongs to $B_p^s(\partial D)$ and that the initial condition g belongs to $B_p^{s+\frac{1}{p}}(D)$, whenever $(s, \frac{1}{p}) \in \mathcal{R}$. The result follows from those by T. Jakab and M. Mitrea in Math. Res. Lett. 13 (2006), no. 5-6, 825–831 and from *Parabolic Besov regularity for the heat equation* by the authors available in *http://www.cimec.org.ar/ojs/index.php/cmm/article/view/3729*

1. INTRODUCTION

In the spirit of the elliptic results by Dahlke and DeVore in [DD], based in [DJP], the improvement of the regularity exponent in Besov norms becomes a tool to measure the rate of convergence of nonlinear approximation methods. A parabolic Besov regularity improvement for temperatures, i.e., solutions of $\frac{\partial u}{\partial t} = \Delta u$, follows from the results obtained by the authors in [AG]. Nevertheless to apply those results, a starting Besov regularity for the temperature is required. In the elliptic case this initial regularity for the harmonic function is proved to be attained if we solve the Dirichlet problem with Besov boundary data, see the results of Jerison and Kenig in [JK]. Its parabolic counterpart is due to Jakab and Mitrea, see [JM].

We shall precisely introduce in the next section all the Besov spaces involved in our main result, the elliptic Besov space $B_p^s(D)$ for a bounded Lipschitz domain in $\mathbb{R}^d$, the boundary elliptic Besov space $B_p^s(\partial D)$ and the parabolic Besov space $\mathbb{B}_p^s(\Omega)$ where $\Omega = D \times (0,T)$.

For a given $0 < \varepsilon \leq 1$ we shall write $\mathcal{R}_\varepsilon$ to denote the set of those points $(a, b) \in [0, 1]^2$ in the plane satisfying any one of the following three conditions

$$\frac{1 - \varepsilon}{2} < b < \frac{1 + \varepsilon}{2} \text{ and } 0 < a < 1;$$

$$\frac{1 + \varepsilon}{2} \leq b < 1 \text{ and } 2b - 1 - \varepsilon < a < 1;$$

$$0 < b \leq \frac{1 - \varepsilon}{2} \text{ and } 0 < a < 2b + \varepsilon.$$

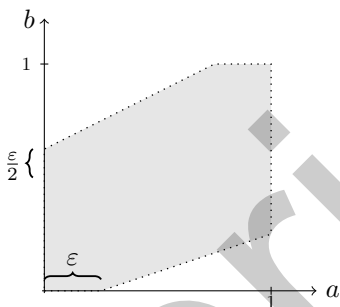The following figure contains a picture of $\mathcal{R}_\varepsilon$ for $\varepsilon = \frac{1}{4}$.

FIGURE 1. The region $\mathcal{R}_\varepsilon$ for the parameters $(s, \frac{1}{p})$ of the regularity of data.

In this note we aim to prove the following result.

Theorem 1. *Let D be a bounded and Lipschitz domain contained in $\mathbb{R}^d$ and let $T > 0$ be given. Let $\Omega = D \times (0, T)$ be the associated parabolic domain. Then there exists a positive number $\varepsilon \leq 1$ depending only on D such that for each p and each s with $(s, \frac{1}{p}) \in \mathcal{R}_\varepsilon$, a solution of the initial-boundary value problem*

$$(P) \qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u, & \text{in } \Omega \\ u(x, t) = f(x), & \text{for } (x, t) \in \partial D \times (0, T) \\ u(x, 0) = g(x), & \text{for } x \in D \end{cases}$$

belongs to the parabolic Besov space $\mathbb{B}_\tau^\alpha(\Omega)$ with $0 < \alpha < \min\{d\frac{p-1}{p}, (s + \frac{1}{p})\frac{d}{d-1}\}$ and $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$ provided that $f \in B_p^s(\partial D)$ and $g \in B_p^{s + \frac{1}{p}}(D)$.

The result becomes relevant when α can be taken larger than $s + \frac{1}{p}$, see Figure 2. The arguments given in [DD] regarding the improvement of the regularity parameter α as a tool to improve the rate of convergence of nonlinear approximation methods for elliptic problems, extend after Theorem 1 to diffusion problems.

2. Proof of Theorem 1

Let us start this section by giving a brief description of the Besov spaces involved. Even when several approaches are possible, we prefer, for the sake of simplicity the interpolation one.
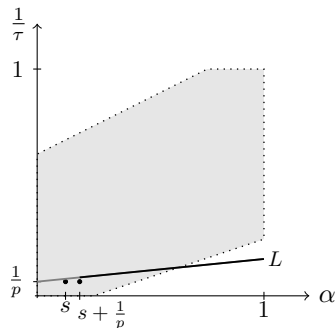
Figure 2. The black segment in the line L shows the improved regularity with data associated to $(s, \frac{1}{p}) \in \mathcal{R}_\varepsilon$ when $d = 10$.

The complete scale of Besov spaces $B_{p,q}^s(\mathbb{R}^d)$ indexed by the parameters s, p, q in the space $\mathbb{R}^d$ is well known and several equivalent versions can be found in the classical literature such as Peetre's book [Pee] and some modern approaches in the book by Y. Meyer [Mey] just to mention two standard references. We shall only deal with the case $p = q$ and we shall write B_p^s instead of $B_{p,p}^s$. Given an open subset D in $\mathbb{R}^d$ one can define $B_p^s(D)$ as the space of all the restrictions to D of the functions in $B_p^s(\mathbb{R}^d)$. A second way to define $B_p^s(D)$ is provided by the real interpolation between Lebesgue and Sobolev spaces. Precisely, for $0 < s < 1$ and $1 \le p \le \infty$, $B_p^s(D) = [L_p(D), W_p^1(D)]_{s,p}$ the s-interpolated between $L_p(D)$ and $W_p^1(D)$. When D is a bounded Lipschitz domain in $\mathbb{R}^d$, both approaches coincide. The initial condition g in our problem (P) belongs to $B_p^{s+\frac{1}{p}}(D)$ in the above described sense.

The second Besov space is involved in the boundary condition f and has to be described by using the local parametrization of the boundary ∂D of D. After the standard localization arguments the problem of defining $B_p^s(\partial D)$, reduces to define the corresponding Besov class on the graph G of a Lipschitz function ϕ with domain in $\mathbb{R}^{d-1}$. This is done by saying that $f \in B_p^s(G)$ when $f(x, \phi(x)) \in B_p^s(\mathbb{R}^{d-1})$ for $0 < s \le 1$ and $p > 0$.

Let us now introduce through interpolation the parabolic Besov spaces involved in the statement of Theorem 1. We would like to point out that Besov scales have been considered in very general settings such as spaces of homogeneous type, see [HS] for example. The approach there is of Littlewood-Paley type. See also [Sch], [ST] and [BIN] for more literature on the subject. For $1 \le p \le \infty$ the anisotropic Sobolev space $W_p^{2,1}(\Omega)$ is defined by the norm

$$\|v\|_{W_p^{2,1}(\Omega)} = \|v\|_{L_p(\Omega)} + \sum_{i=1}^d \left\|\frac{\partial v}{\partial x_i}\right\|_{L_p(\Omega)} + \sum_{i=1}^d \sum_{j=1}^d \left\|\frac{\partial^2 v}{\partial x_i \partial x_j}\right\|_{L_p(\Omega)} + \left\|\frac{\partial v}{\partial t}\right\|_{L_p(\Omega)}.$$

For $0 < \alpha < 2$ we define

$$B_p^{\alpha, \frac{\alpha}{2}}(\Omega) = [L_p(\Omega), W_p^{2,1}(\Omega)]_{\frac{\alpha}{2}, p},$$

the $\frac{\alpha}{2}$-real interpolated space between $W_p^{0,0}(\Omega) = L_p(\Omega)$ and $W_p^{2,1}(\Omega)$. For simplicity we introduce the notation $\mathbb{B}_p^\alpha(\Omega)$ for the space $B_p^{\alpha,\frac{\alpha}{2}}(\Omega)$.

Before proving Theorem 1 we shall introduce two known regularity results for solutions of the heat equation. The first one, due to Jakab and Mitrea [JM], gives the *preservation* of Besov regularity from boundary and source data in the solution of the heat equation with homogeneous initial data. The second, contained in [AG], provides the *improvement* of Besov regularity for the case of vanishing source.

Theorem 2. *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$ ($d \geq 2$) and let $T > 0$ be given. Set $\Omega = D \times (0,T)$.*

(A) ([JM]) There exists $\varepsilon > 0$ depending only on the Lipschitz character of D such that for every $(s, \frac{1}{p}) \in \mathcal{R}_\varepsilon$ the solution of

(P_1)
$$\begin{cases} (\frac{\partial}{\partial t} - \Delta)v = \varphi, & in\ \Omega \\ v = \psi, & in\ \partial D \times (0,T) \\ v(x,0) = 0, & for\ x \in D \end{cases}$$

belongs to $L_p((0,T); B_p^{s+\frac{1}{p}}(D))$ provided that $\psi \in B_p^s(\partial D)$ and $\varphi \in B_p^{s+\frac{1}{p}-2}(D)$.

(B) ([AG]) For $1 < p < \infty$, $\lambda > 0$ and u a solution of
$$\frac{\partial u}{\partial t} = \Delta u \quad in\ \Omega$$

with $u \in L_p((0,T); B_p^\lambda(D))$, we have that $u \in \mathbb{B}_\tau^\alpha(\Omega)$ with $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$ and $0 < \alpha < \min\{d\frac{p-1}{p}, \lambda\frac{d}{d-1}\}$.

Part (A) of Theorem 2 follows from Theorem (1.1) and equation (5) in [JM]. Part (B) is obtained in [AG] which is itself the third part of the parabolic Besov regularity improvement program started in [AGI2] and [AGI1].

Proof of Theorem 1. Let u be a solution of (P) then $v = u - g$ is a solution of (P_1) with $\varphi = -\Delta g$ and $\psi = f - g$. Now since $g \in B_p^{s+\frac{1}{p}}(D)$ we have that $\varphi = -\Delta g \in B_p^{s+\frac{1}{p}-2}(D)$. On the other hand from Theorem 3.1 in [JK] we know that the trace of g on ∂D belongs to $B_p^s(\partial D)$ since $g \in B_p^{s+\frac{1}{p}}(D)$. Hence $\psi \in B_p^s(\partial D)$. We are in position to apply part (A) of Theorem 2 to obtain that $v \in L_p((0,T); B_p^{s+\frac{1}{p}}(D))$.

Since g belongs to $B_p^{s+\frac{1}{p}}(D)$ and T is finite we have that $g \in L_p((0,T); B_p^{s+\frac{1}{p}}(D))$. Next we apply part (B) of Theorem 2 with $\lambda = s + \frac{1}{p}$ and we get that $u \in \mathbb{B}_\tau^\alpha(\Omega)$ with $0 < \alpha < \min\{d\frac{p-1}{p}, \lambda\frac{d}{d-1}\}$ and $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$ as desired. $\qquad\square$

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Instituto de Matemática Aplicada del Litoral (CONICET, UNL)
Departamento de Matemática, Facultad de Ingeniería Química (UNL)

Güemes 3450
S3000GLN Santa Fe
Argentina.

E-mail address: haimar@santafe-conicet.gov.ar

E-mail address: ivanagomez@santafe-conicet.gov.ar