Weighted local BMO spaces and the local Hardy-Littlewood maximal Operator

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1 Introduction

Let us fix $\kappa > 1$. We say that $I = (a, b)$ is a $\kappa-$**local interval** whenever $0 < a < b < \kappa a$ and we will call **critical intervals** to those of the form $(a, \kappa a)$ for $a > 0$. Also we shall denote with $\mathcal{I}_\kappa$ the family of all local intervals with respect to κ. With this notation we introduce the definition of the κ-local Maximal Operator on $\mathbb{R}^+ = (0, \infty)$ as follows: Given any measurable function $f : \mathbb{R}^+ \to \mathbb{R}$, we set

$$\mathrm{M}^\kappa_{loc} f(x) = \sup_{x \in I \in \mathcal{I}_\kappa} \frac{1}{|I|} \int_I |f(y)| dy,$$

for any $x \in \mathbb{R}^+$. This operator, being smaller than the regular Hardy-Littlewood maximal function, is bounded on Lebesgue spaces $L^p(\mathbb{R}^+)$ for $1 < p < \infty$ and of weak type for $p = 1$. However, as it was shown in [4], L^p-weighted inequalities hold for a wider class than Muckenhoupt's A_p weigths, the $A^p_{loc,\kappa}$ classes, which require control of averages only for local intervals. Nowak and Stempak studied this problem in connection with transplantation theorems associated to Hankel Transforms. Such classes of weights were also used in [2] to prove weighted inequalities for the maximal operator of the diffusion semigroup associated to Laguerre functions systems.

To be precise, we call a **weight** on $\mathbb{R}^+$ to any nonnegative and $\mathbb{R}^+$-locally integrable function. We shall denote by $A^p_{loc,\kappa}$ the class of all weights ω on $\mathbb{R}^+$ such that there exists $C_\kappa > 0$ satisfying, for all $I \in \mathcal{I}_\kappa$,

$$\left(\int_I \omega(x) dx \right)^{1/p} \left(\int_I \omega(x)^{-p'/p} dx \right)^{1/p'} \leq C_\kappa |I|, \tag{1.1}$$

when $p > 1$, and

$$\omega(I) \leq C_\kappa |I| \inf_{x \in I} \omega(x), \tag{1.2}$$

when $p = 1$.

The semi-norm $[\omega]_{p,\kappa}$ is the least constant C for which (1.1) or (1.2) holds.

In [4], the authors prove that, for a fixed p, the classes are independent of κ, namely, that $A^p_{loc,\kappa} = A^p_{loc,2}$, for any $\kappa > 1$. Therefore we will denote that class just with A^p_{loc}. Nevertheless let us observe that the semi-norms $[\omega]_{p,\kappa}$ actually depend on κ and could happen, for some weight $\omega \in A^p_{loc}$, that $[\omega]_{p,\kappa} \to \infty$ as $\kappa \to \infty$. Such is the case of $\omega(x) = 1/x$.

In the same article, the authors also show that M^κ_{loc} is bounded on $L^p(\omega)$ if and only if $\omega \in A^p_{loc}$, for $1 < p < \infty$, and that M^κ_{loc} is of weak type $(1,1)$ with respect to $\omega(x)dx$ if and only if $\omega \in A^1_{loc}$, with boundedness constants depending on κ only by $[\omega]_{p,\kappa}$.

On the other hand, it is well known that M, the usual Hardy-Littlewood maximal operator, is not bounded on BMO, the space of John and Nirenberg. In fact, it was shown in [1] that for a $BMO(\mathbb{R}^n)$ function, $M(f)$ is either identically ∞ or it does belong to BMO. They also show that if the underlying space is a cube then M is actually bounded on BMO.

The purpose of this work is to investigate the behavior of the Local Maximal Operator on appropriate weighted BMO spaces. We believe that our result (see theorem below) is new even in the unweighted case.

2 Some preliminary results

From their definition it is clear that A^p_{loc} classes satisfy

1. $A^p_{loc} \subset A^q_{loc}, 1 \leq p \leq q$;
2. $\omega \in A^p_{loc}$ if and only if $\omega^{1-p'} \in A^{p'}_{loc}$.

As usual, we denote $A^\infty_{loc} = \bigcup_{p\geq 1} A^p_{loc}$. The following property, that we borrow from [4], will be useful in the sequel.

Lemma 2.1. *Let $\omega \in A^p_{loc}$, $1 \leq p < \infty$. Then, for every $\kappa > 1$, there exists a constant C_κ depending on κ, p and the A^p_{loc}-norm of ω, such that*

$$\frac{\omega(I)}{\omega(S)} \leq C_\kappa \left(\frac{|I|}{|S|}\right)^p,$$

for any $I \in \mathcal{I}_\kappa$ and any measurable set $S \subset I$.

Remark 2.2. The case $p = 1$ of this Lemma arises directly from A^1_{loc} class definition, since for all $S \subset I$ we have $\inf_{x\in I} \omega(x) \leq \inf_{x\in S} \omega(x) \leq |S|\omega(S)$. Thus, (1.2) give us

$$\omega(I) \leq C_\kappa \frac{|I|}{|S|}\omega(S) \tag{2.1}$$

for any $I \in \mathcal{I}_\kappa$ and any set $S \subset I$.

Next we introduce the precise definitions of the Hardy-Littlewood maximal operator supported on a given cube and the corresponding BMO space.

Let Q a fixed cube in $\mathbb{R}^n$. The Hardy-Littlewood maximal function M_Q supported on Q is given, for any Q-locally integrable function f and any $x \in Q$, by

$$M_Q f(x) = \sup \frac{1}{|Q'|}\int_{Q'} |f(y)|dy,$$

where the supremum is taken over all cubes Q' contained in Q and containing x.

Given a weight ω defined on Q, the weighted Bounded Mean Oscillation Space on Q, $BMO_Q(\omega)$, is defined as the set of Q-locally integrable functions f that satisfies

$$\frac{1}{\omega(I)}\int_I |f(x) - f_I|dx \leq C, \tag{2.2}$$

for all cubes $I \subset Q$, where $f_I = \frac{1}{|I|}\int_I f(x)dx$. The semi-norm $\|f\|_{BMO_Q(\omega)}$ is the least constant C that satisfies this condition.

With $BMO_n(\omega)$ we denote the space when $Q = \mathbb{R}^n$ andd in that case we required f to be locally integrable and satisfying (2.2) for any cube $I \subset \mathbb{R}^n$.

In [1], the unweighted version of the following result is established (see theorem 4.2 there). We claim that the same proof, with some obvious modifications, can be adapted to this setting.

Theorem 2.3.

1. *Let Q a fixed cube in $\mathbb{R}^n$ and ω a weight of $A^1(Q)$ class. If f belongs to $BMO_Q(\omega)$ then $M_Q f$ belongs to $BMO_Q(\omega)$ and*

$$\|M_Q f\|_{BMO_Q(\omega)} \leq C\|f\|_{BMO_Q(\omega)},$$

where C depends only on the dimension n and the $A^1(Q)$ constant of ω.

2. *Let $\omega \in A^1(\mathbb{R}^n)$. If f belongs to $BMO_n(\omega)$ and if Mf is not identically infinity, then Mf belongs to $BMO_n(\omega)$ and*

$$\|Mf\|_{BMO_n(\omega)} \leq C\|f\|_{BMO_n(\omega)},$$

where C depends only on the dimension n and the A^1 constant of ω.

3 Local BMO space

For $\kappa > 1$ and a $\mathbb{R}^+$ weight ω, we denote with $BMO^{\kappa}_{loc}(\omega)$ the family of all functions $f \in L^1_{loc}(\mathbb{R}^+)$ that satisfy the *bounded oscillation condition*

$$\frac{1}{\omega(I)} \int_I |f(x) - f_I| dx \leq C_\kappa, \quad \text{for all } I \in \mathcal{I}_\kappa, \tag{3.1}$$

and the *bounded mean condition*

$$\frac{1}{\omega(I)} \int_I |f(x)| dx \leq C_\kappa, \quad \text{for all } I \in \mathcal{I}^c_\kappa, \tag{3.2}$$

where we consider $\mathcal{I}^c_\kappa = \{(a,b) : a > 0, b \geq \kappa a\}$.

The $BMO^{\kappa}_{loc}(\omega)$ norm of f is the least constant that satisfies both conditions and will be denoted with $\|f\|_{BMO^{\kappa}_{loc}(\omega)}$.

Observe that, since $\frac{1}{\omega(I)} \int_I |f(x) - f_I| dx \leq 2\frac{1}{\omega(I)} \int_I |f(x)| dx$ for any measurable set I, we have that the bounded oscillation condition (3.1) actually holds for any interval $I \subset\subset \mathbb{R}^+$. Also, if $1 < \kappa < \kappa'$ then $BMO^{k}_{loc}(\omega) \hookrightarrow BMO^{k'}_{loc}(\omega)$. Moreover, we have:

Lemma 3.1. *If $\omega \in A^\infty_{loc}$ then $BMO^{k}_{loc}(\omega) = BMO^{k'}_{loc}(\omega)$ for any $\kappa, \kappa' > 1$, with norms and equivalence constants depending on ω, κ and κ'.*

Proof. Consider $1 < \kappa < \kappa'$. By the observation made before, it will be enough to prove, for $f \in BMO^{\kappa'}_{loc}(\omega)$ and $I \in \mathcal{I}^c_\kappa$, that $\frac{1}{\omega(I)} \int_I |f(x)| dx \leq C\|f\|_{BMO^{\kappa'}_{loc}}$, with $C = C(\kappa, \kappa', \omega)$.

If $I \in \mathcal{I}^c_{k'}$, then there is nothing to prove. If $I = (a,b) \in \mathcal{I}_{k'} \cap \mathcal{I}^c_\kappa$ we have $\kappa a \leq b < \kappa' a$. Then, using Lemma 2.1, we obtain

$$\begin{aligned}
\int_I |f(x)| dx &\leq \int_a^{\kappa' a} |f(x)| dx \\
&\leq \|f\|_{BMO^{\kappa'}_{loc}} \omega((a, \kappa' a)) \\
&\leq C\|f\|_{BMO^{\kappa'}_{loc}} \omega((a, \kappa a)) \\
&\leq C\|f\|_{BMO^{\kappa'}_{loc}} \omega(I),
\end{aligned}$$

and this complete the proof.

□

The following Lemma says that is enough to prove the bounded mean condition (3.2) only for critical intervals to conclude that a function is in $BMO^{\kappa}_{loc}(\omega)$.

Lemma 3.2. *Let $\omega \in A^\infty_{loc}$. If a function f satisfies (3.2) for any $I = (a, \kappa a)$ with $a > 0$, then f satisfies (3.2) for any $I \in \mathcal{I}^c_\kappa$.*

Proof. Let $I = (a, b)$ with $b > \kappa a$ and let $j_0 \geq 1$ an integer such that $\kappa^{j_0} a < b \leq \kappa^{j_0+1} a$. Then

$$\int_I |f(x)|dx \leq \sum_{j=0}^{j_0} \int_{\kappa^j a}^{\kappa^{j+1} a} |f(x)|dx.$$

Since each $(\kappa^j a, \kappa^{j+1} a)$ are critical intervals, by hypothesis we have

$$\begin{aligned} \int_I |f(x)|dx &\leq C_\kappa \sum_{j=0}^{j_0} \omega\left((\kappa^j a, \kappa^{j+1} a)\right) \\ &\leq C_\kappa \left[\omega(I) + \omega\left((\kappa^{j_0} a, \kappa^{j_0+1} a)\right)\right]. \end{aligned}$$

Since the interval $(\kappa^{j_0-1} a, \kappa^{j_0+1} a)$ belongs to $\mathcal{I}_{\kappa^3}$, Lemma 2.1 implies

$$\begin{aligned} \omega\left((\kappa^{j_0} a, \kappa^{j_0+1} a)\right) &\leq \omega\left((\kappa^{j_0-1} a, \kappa^{j_0+1} a)\right) \\ &\leq C_\kappa\ \omega((\kappa^{j_0-1} a, \kappa^{j_0} a)) \\ &\leq C_\kappa \omega(I). \end{aligned}$$

Thus, we have obtained $\int_I |f(x)|dx \leq C_\kappa\ \omega(I)$ for any $I \in \mathcal{I}^c_\kappa$. □

Another useful property is the following one. Note that, in the classic BMO context, this is a consequence of John-Nirenberg inequality.

Lemma 3.3. *Equivalence of norm's property. Let $\omega \in A^p_{loc}$ and $\kappa > 1$. For $1 \leq r \leq p'$, there exists a constant $C_\kappa = C(r, \kappa, [\omega]_{p,\kappa})$ such that if $f \in BMO^\kappa_{loc}(\omega)$ then*

$$\left(\frac{1}{\omega(I)} \int_I |f(x) - f_I|^r \omega^{1-r}(x)dx\right)^{1/r} \leq C_\kappa\ \|f\|_{BMO^\kappa_{loc}(\omega)} \tag{3.3}$$

for all $I \in \mathcal{I}_\kappa$, and

$$\left(\frac{1}{\omega(I)} \int_I |f(x)|^r \omega^{1-r}(x)dx\right)^{1/r} \leq C_\kappa,\ \|f\|_{BMO^\kappa_{loc}(\omega)} \tag{3.4}$$

for all $I \in \mathcal{I}^c_\kappa$.

Proof. Let $\omega \in A^p_{loc}$ and $f \in BMO^\kappa_{loc}(\omega)$. First, we will prove that (3.3) holds. For any $i \in \mathbb{Z}$, let $J_i = (\kappa^i, \kappa^{i+3})$. Then, $\omega \in A^p_{loc}$ and $J_i \in \mathcal{I}_{\kappa^4}$ implies $\omega \in A^p(J_i)$, with $[\omega]_{A^p(J_i)} \leq [\omega]_{p,\kappa}$, for any $i \in \mathbb{Z}$.

Since $BMO^\kappa_{loc}(\omega) \subset BMO(\omega)$, the BMO space supported on $\mathbb{R}^+$, $f|_{J_i} \in BMO_{J_i}(\omega)$ and from the known equivalence of norm's inequality for $BMO_{J_i}(\omega)$ we have

$$\left(\frac{1}{\omega(I)}\int_I |f(x) - f_I|^r \omega^{1-r}(x)dx\right)^{1/r} \leq C_i \|f\|_{BMO^\kappa_{loc}(\omega)},$$

for any $I \subset J_i$. Since the constant C_i depend of i only by $[\omega]_{A^p(J_i)}$, we can replace it by a constant C_κ independent of J_i. Thus, since every $I \in \mathcal{I}_\kappa$ is contained in some J_i, $i \in \mathbb{Z}$, we obtain the desired result (3.3).

To prove that (3.4) holds for $I = (a, \kappa a)$, observe that

$$\begin{aligned}\left(\frac{1}{\omega(I)}\int_I |f(x)|^r \omega^{1-r}(x)dx\right)^{1/r} &\leq \left(\frac{1}{\omega(I)}\int_I |f(x) - f_I|^r \omega^{1-r}(x)dx\right)^{1/r} \\ &+ \left(\frac{\omega^{1-r}(I)}{\omega(I)}\right)^{1/r} |f_I|.\end{aligned}$$

The first term of the right side is bounded by $\|f\|_{BMO^\kappa_{loc}(\omega)}$, we can prove this following the same argument as is the proof of (3.3). For the second term, observe that I belonging to $\mathcal{I}_{\kappa^2}$ and ω^{1-r} belonging to A^r_{loc} imply that $\omega^{1-r}(I)^{1/r}\omega(I)^{1/r'} \leq C_\kappa\, |I|$. Then

$$\begin{aligned}\left(\frac{\omega^{1-r}(I)}{\omega(I)}\right)^{1/r} |f_I| &\leq C_\kappa \frac{1}{\omega(I)}\int_I |f(x)|dx \\ &\leq C_\kappa\, \|f\|_{BMO^\kappa_{loc}(\omega)}.\end{aligned}$$

To extend this result to intervals $I = (a, b)$ with $b > \kappa a$, we proceed as we did in the proof of Lemma 3.2. □

We now state our main result.

Theorem 3.4. *If $\kappa > 1$ and $\omega \in A^1_{loc}$ then there exist a constant $C = C(\kappa, [\omega]_{1,\kappa})$ such that*

$$\|\mathrm{M}^\kappa_{loc} f\|_{BMO^\kappa_{loc}(\omega)} \leq C \|f\|_{BMO^\kappa_{loc}(\omega)}$$

for all $f \in BMO^\kappa_{loc}(\omega)$.

Proof. Let $f \in BMO^{\kappa}_{loc}(\omega)$.

We will prove first that the bounded oscillation condition (3.1) holds for $\mathrm{M}^{\kappa}_{loc} f$. Consider $I = (a, b) \in \mathcal{I}_{\kappa}$, i.e., $0 < a < b < \kappa a$. We want to prove

$$\frac{1}{\omega(I)} \int_I |\mathrm{M}^{\kappa}_{loc} f(x) - c| dx \leq C \|f\|_{BMO^{\kappa}_{loc}(\omega)}, \tag{3.5}$$

for some constant c depending on f and I and $C = C(\kappa, [\omega]_{1,\kappa})$.

Let $j_0 \in \mathbb{Z}$ such that $\kappa^{j_0} < a \leq \kappa^{j_0+1}$ and call $I_0 = (\kappa^{j_0-1}, \kappa^{j_0+3})$. Then, for any $x \in I$ and any $J = (a', b') \in \mathcal{I}_{\kappa}$ with $x \in J$, we have $J \subset I_0$. That is true since $I \cap J \neq \emptyset$ implies $a' < b$ and $b' > a$ and therefore $b' < \kappa a' < \kappa b < \kappa^2 a \leq \kappa^{j_0+3}$ and $a' > b'/\kappa > a/\kappa > \kappa^{j_0-1}$. Then, for any $x \in I$,

$$\mathrm{M}^{\kappa}_{loc} f(x) \leq M_{I_0} f(x),$$

where M_{I_0} is the Hardy Littlewood maximal operator supported in I_0. That is, for $x \in I$ we take averages only over intervals contained in I_0.

We bound the left side of (3.5) by the sum of A and B, where

$$A = \frac{1}{\omega(I)} \int_I |\mathrm{M}^{\kappa}_{loc} f(x) - M_{I_0} f(x)| dx$$

and

$$B = \frac{1}{\omega(I)} \int_I |M_{I_0} f(x) - c| dx.$$

We first consider A. Since for all $x \in I$ we have $\mathrm{M}^{\kappa}_{loc} f(x) \leq M_{I_0} f(x) \leq \mathrm{M}^{\kappa}_{loc} f(x) + \widetilde{\mathrm{M}}^{\kappa}_{loc} f(x)$,

where

$$\widetilde{\mathrm{M}}^{\kappa}_{loc} f(x) = \sup_{x \in J \subset I_0, J \in \mathcal{I}^c_{\kappa}} \frac{1}{|J|} \int_J |f(y)| dy,$$

we have

$$A \leq \frac{1}{\omega(I)} \int_I \widetilde{\mathrm{M}}^{\kappa}_{loc} f(x) dx.$$

If $J \in \mathcal{I}^c_{\kappa} = \{(\alpha, \beta) : \alpha > 0, \beta \geq \kappa \alpha\}$ and $J \subset I_0 = (\kappa^{j_0-1}, \kappa^{j_0+3})$, then $|J| > (\kappa - 1)\kappa^{j_0-1} = \frac{\kappa-1}{\kappa^4-1} |I_0|$. This implies that

$$\begin{aligned} \widetilde{\mathrm{M}}^{\kappa}_{loc} f(x) &\leq C_{\kappa} \frac{1}{|I_0|} \int_{I_0} |f(y)| dy \\ &\leq C_{\kappa} \|f\|_{BMO^{\kappa}_{loc}(\omega)} \frac{\omega(I_0)}{|I_0|}, \end{aligned}$$

for any $x \in I$, where the last inequality arises from (3.2) since $f \in BMO^{\kappa}_{loc}(\omega)$ and $I_0 \in \mathcal{I}^c_{\kappa}$.

Then

$$A \leq C_\kappa \, \|f\|_{BMO^\kappa_{loc}(\omega)} \frac{|I|}{\omega(I)} \frac{\omega(I_0)}{|I_0|}. \tag{3.6}$$

Since $\omega \in A^1_{loc}$, $I_0 \in \mathcal{I}_{\kappa^5}$ and $I \subset I_0$, (2.1) implies

$$\omega(I_0) \leq C_\kappa \frac{|I_0|}{|I|} \omega(I)$$

and then A is bounded by $\|f\|_{BMO^\kappa_{loc}(\omega)}$ times a constant $C = C(\kappa, [\omega]_{1,\kappa})$.

In order to obtain the same for

$$B = \frac{1}{\omega(I)} \int_I |M_{I_0} f(x) - c| dx,$$

consider $c = (M_{I_0} f)_I$. Observe that $M_{I_0} f < \infty$ *a.e.*, since $f \in BMO(\omega)$. Also, $\omega \in A^1_{loc}$ and $I_0 \in \mathcal{I}_{\kappa^5}$ implies $\omega \in A^1(I_0)$, with the $A^1(I_0)$ constant depending only of $[\omega]_{1,\kappa^5}$, that is, independent of I_0 and hence of I. Then, we use Theorem 2.3 with $Q = I_0$ to obtain $B \leq C\|f_0\|_{BMO_{I_0}(\omega)}$, with $C = C(\kappa, \omega)$. Since $\|f_0\|_{BMO_{I_0}(\omega)} \leq \|f\|_{BMO(\omega)} \leq \|f\|_{BMO^\kappa_{loc}(\omega)}$, we obtain the desired inequality.

Now we will prove that the bounded mean condition (3.2) for $\mathrm{M}^\kappa_{loc} f$. By Lemma 3.2, it will be enough to prove

$$\frac{1}{\omega(I)} \int_I |\mathrm{M}^\kappa_{loc} f(x)| dx \leq C_\kappa \|f\|_{BMO^\kappa_{loc}(\omega)} \tag{3.7}$$

for $I = (a, \kappa a)$, where $a > 0$.

Let $I^* = (a/\kappa, (\kappa+1)a)$ and write $f = f_1 + f_2$, where $f_1 = f\chi_{I^*}$ and $f_2 = f\chi_{I^{*c}}$, where the complement is taken on $\mathbb{R}^+$. We will prove (3.7) for $\mathrm{M}^\kappa_{loc} f_1$ and $\mathrm{M}^\kappa_{loc} f_2$ separately.

Consider first $\mathrm{M}^\kappa_{loc} f_1$. From Hölder inequality we have

$$\frac{1}{\omega(I)} \int_I |\mathrm{M}^\kappa_{loc} f_1(x)| dx \leq \left(\frac{1}{\omega(I)} \int_I |\mathrm{M}^\kappa_{loc} f_1(x)|^2 \omega^{-1}(x) dx \right)^{1/2}. \tag{3.8}$$

Since $\omega \in A^1_{loc} \subset A^2_{loc}$ and hence $\omega^{-1} \in A^2_{loc}$, we have, by Proposition 6.3 of [4], that M^κ_{loc} is of strong type $(2, 2)$ with weight ω^{-1}. Then, the right side of (3.8) is bounded by a constant times

$$\left(\frac{1}{\omega(I)} \int_I |f_1(x)|^2 \omega^{-1}(x) dx \right)^{1/2}. \tag{3.9}$$

Since $I^* \in \mathcal{I}_{2\kappa^2}$ and $I \subset I^*$, (2.1) implies $\omega(I^*) \le C_\kappa\, \omega(I)$, and then (3.9) is bounded by

$$C_\kappa \left(\frac{1}{\omega(I^*)} \int_{I^*} |f(x)|^2 \omega^{-1}(x) dx \right)^{1/2}.$$

Finally, since $I^* \in \mathcal{I}_\kappa^c$, we use the equivalence of norm's inequality (3.4) with $r = 2$ and we obtain that the left side of (3.8) is bounded by a constant $C = C([\omega]_{1,\kappa}, \kappa)$ times $\|f\|_{BMO^\kappa_{loc}(\omega)}$.

Consider now $\mathrm{M}^\kappa_{loc} f_2(x)$, with $x \in I = (a, \kappa a)$. Let us observe that here is enought to take the supremum of the averages over those $J \in \mathcal{I}_\kappa$ such that $x \in J$ and $J \cap I^{*c} \neq \emptyset$. Remember that $I^* = (\frac{a}{\kappa}, (\kappa+1)a)$. If an interval $J = (a', b')$ satisfies $J \cap I \neq \emptyset$, then $a' < \kappa a$ and $a < b'$. If it also $J \in \mathcal{I}_\kappa$, then $a' > a/\kappa$ and $b' < \kappa^2 a$. Then we have $J \subset I^{**}$, where $I^{**} \doteq (a/\kappa, \kappa^2 a)$. Also, if $J \cap I^{*c} \neq \emptyset$ then $b' \ge (\kappa+1)a$ and this, together with $a' < \kappa a$, implies $|J| > C_\kappa |I^{**}|$. Thus, for every $x \in I$ we have

and

$$\begin{aligned} \mathrm{M}^\kappa_{loc} f_2(x) &\le C_\kappa \frac{1}{|I^{**}|} \int_{I^{**}} |f(y)| dy \\ &\le C_\kappa\, \|f\|_{BMO^\kappa_{loc}(\omega)} \frac{\omega(I^{**})}{|I^{**}|}, \end{aligned} \tag{3.10}$$

where the last inequality arises since $f \in BMO^\kappa_{loc}(\omega)$ and $I^{**} \in \mathcal{I}^c_\kappa$. Finally, since $\omega \in A^1_{loc}$, $I^{**} \in \mathcal{I}_{\kappa^4}$ and $I \subset I^{**}$, (2.1) implies

$$\frac{1}{\omega(I)} \int_I |\mathrm{M}^\kappa_{loc} f_2(x)| dx \le C_\kappa\, \|f\|_{BMO^\kappa_{loc}(\omega)}.$$

Therefore, the proof of Theorem 3.4 is complete.

□

4 A necessary condition.

In [3], Muckenhoupt and Wheeden introduced another version of weighted BMO. More precisely, for a given interval I, $\omega(I)$ is replaced by $\inf_{x \in I} \omega(x)|I|$. Similarly, we consider now the corresponding local version $BMO^{\kappa,*}_{loc}(\omega)$, the space of all $\mathbb{R}^+$-locally integrable function f that satisfy

$$\frac{1}{\inf_{x\in I} \omega(x)|I|} \int_I |f(x) - f_I| dx \le C_\kappa, \quad \text{for all } I \in \mathcal{I}_\kappa, \tag{4.1}$$

and

$$\frac{1}{\inf_{x\in I}\omega(x)|I|}\int_I |f(x)|dx \le C_\kappa, \quad \text{for } I=(a,\kappa a), a>0, \tag{4.2}$$

and the norm $\|f\|_{BMO^{\kappa,*}_{loc}(\omega)}$ will be the least constant satisfying both conditions.

It is clear that $BMO^{\kappa,*}_{loc}(\omega)\subset BMO^{\kappa}_{loc}(\omega)$, since for any weight ω and any ball I we have $\omega(B)\ge \inf_{x\in I}\omega(x)|I|$, and, by Lemma 3.2, a function need to satisfy the bounded mean condition only for critical balls in order to be in $BMO^{\kappa}_{loc}(\omega)$. Also, if we suppose $\omega\in A^1_{loc}$, then $BMO^{\kappa,*}_{loc}(\omega)=BMO^{\kappa}_{loc}(\omega)$, with equivalence of norms. Thus, from Theorem 3.4, we have that $\mathrm{M}^{\kappa}_{loc}$ is bounded from $BMO^{\kappa}_{loc}(\omega)$ to $BMO^{\kappa,*}_{loc}(\omega)$, if $\omega\in A^1_{loc}$. We will see now that the converse statement also holds.

Theorem 4.1. *If $\kappa>1$, then $\mathrm{M}^{\kappa}_{loc}: BMO^{\kappa}_{loc}(\omega)\longrightarrow BMO^{\kappa,*}_{loc}(\omega)$ if and only if $\omega\in A^1_{loc}$.*

Proof. From the above remark we only need to prove the necessity of $\omega\in A^1_{loc}$. Suppose then that $\mathrm{M}^{\kappa}_{loc}$ is bounded from $BMO^{\kappa}_{loc}(\omega)$ into $BMO^{\kappa,*}_{loc}(\omega)$ and consider an interval $I\in\mathcal{I}_\kappa$. Since $L^\infty(\omega^{-1})=\{f: f\omega^{-1}\in L^\infty(\mathbb{R}^+)\}$ is continuously contained in $BMO^{\kappa}_{loc}(\omega)$, we have

$$\frac{1}{\inf_{x\in I}\omega(x)|I|^2}\int_I\int_I |\mathrm{M}^{\kappa}_{loc}f(x)-\mathrm{M}^{\kappa}_{loc}f(y)|dxdy \le C\|f\omega^{-1}\|_\infty, \tag{4.3}$$

for every $f\in L^\infty(\omega^{-1})$.

We divide the interval I into six disjoint subintervals of equal measure, that is,

$$I=I_1\cup I_2\cup I_3\cup I_4\cup I_5\cup I_6$$

where all the I_i are disjoint and $|I_i|=\frac{|I|}{6}$. More precisely, if $I=(a,b)$ then $I_i=(a+\frac{b-a}{6}(i-1), a+\frac{b-a}{6}i)$, $i=1,...,6$.

If we take $f=\omega\chi_{I_1}$, then from (4.3) we get

$$\int_{I_4}\int_{I_1}|\mathrm{M}^{\kappa}_{loc}f(x)-\mathrm{M}^{\kappa}_{loc}f(y)|dxdy \le C|I|^2\inf_{x\in I}\omega(x), \tag{4.4}$$

If $x\in I_1$ then clearly $\mathrm{M}^{\kappa}_{loc}f(x)\ge \frac{1}{|I_1|}\int_{I_1}|f|=\frac{\omega(I_1)}{I_1}$. If $y\in I_4$ then for any interval J such that $y\in J$ and $J\cap I_1\neq\emptyset$ we have $|J|>|I_2\cup I_3|=2|I_1|$, thus $\mathrm{M}^{\kappa}_{loc}f(y)\le\frac{1}{2}\frac{\omega(I_1)}{I_1}$. Then we have $|\mathrm{M}^{\kappa}_{loc}f(x)-\mathrm{M}^{\kappa}_{loc}f(y)|\ge C\omega(I_1)/|I|$ for any $x\in I_1$ and $y\in I_4$. So, if we integrate over I_1 and I_4, (4.4) give us

$$\omega(I_1)\le C|I|\inf_{x\in I}\omega(x).$$

Analogously, we can obtain the same inequality for the other intervals I_i, $i = 2, ..., 6$, considering $f = \omega\chi_{I_i}$ and integrating x over I_i and y over I_j, where I_j is at least at a distance $|I|/3$ away from I_i. For example, we may compare I_2 with I_5 and I_3 with I_1 or I_6 and so on.

In this way we will arrive to

$$\omega(I_i) \leq C|I| \inf_{x\in I} \omega(x), \quad \text{for } i = 1, ..., 6.$$

Finally, adding on i we obtain the A^1 condition for the interval $I \in \mathcal{I}_\kappa$. Therefore, $\omega \in A^1_{loc}$.

□

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