Balls as subspaces of homogeneous type: on a construction due to R. Macías and C. Segovia

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1 Introduction

Let Q_0 be a cube in $\mathbf{R}^n$. Let Q be another cube in $\mathbf{R}^n$, with sides parallel to those of Q_0, centered at a point of Q_0 and side length $l(Q)$ less than that of Q_0, then the intersection of both cubes contains another cube with side length at least $\frac{l(Q)}{2^n}$.

This geometric property is actually valid for the family of balls defined by a norm in $\mathbf{R}^n$. Not just for $\|x\|_\infty = \sup\{|x_1|, \cdots, |x_n|\}$, for which the balls are the cubes. In fact, take a point x_0 in the Euclidean space $\mathbf{R}^n$ and a positive number R. Consider the Euclidean distance $|x-y|$, or any other norm in $\mathbf{R}^n$. Set $B(x_0, R) = \{x \in \mathbf{R}^n : |x - x_0| < R\}$. Pick a point $x \in B(x_0, R)$ and $0 < r < R$. Then, $B(x + \frac{r}{2}\frac{x_0-x}{|x-x_0|}, r/2) \subset B(x_0, R) \cap B(x, r)$. In fact, for $y \in B(x + \frac{r}{2}\frac{x_0-x}{|x-x_0|}, \frac{r}{2})$ we have that $|y-x| \leq \left|y - x - \frac{r}{2}\frac{x_0-x}{|x-x_0|}\right| + \frac{r}{2} < r$. In other words $y \in B(x,r)$. Also $|y-x_0| \leq \left|y-x-\frac{r}{2}\frac{x_0-x}{|x-x_0|}\right| + \left|x-x_0+\frac{r}{2}\frac{x_0-x}{|x-x_0|}\right| < \frac{r}{2} + \left|\frac{x-x_0}{|x-x_0|}\left(|x-x_0| - \frac{r}{2}\right)\right| = |x-x_0| < R$, then $y \in B(x_0, R)$. Hence we have that $|B(x_0,R) \cap B(x,r)| \geq a_n r^n$, where a_n depends only on the dimension $n, 0 < r < R$. Here $|E|$ is Lebesgue measure of the set E. On the other hand $|B(x_0,R) \cap B(x,r)| \leq |B(x,r)| = w_n r^n$. In other words, there exists constants c_1 and c_2 such that the inequalities

$$c_1 r^n \leq |B(x,r) \cap B(x_0,R)| \leq c_2 r^n$$

hold for every $x \in B(x_0, R)$ and every $0 < r < R$.

The above property can be stated saying that $\{(B(x_0,R), d, \mu) : x_0 \in \mathbf{R}^n,\ R > 0\}$ is a uniform family of n-normal spaces, where d is the restriction of the Euclidean distance to $B(x_0,R)$ and μ is the restriction of Lebesgue measure. Let us point out that s-normality of the balls of a given distance implies that the Hausdorff dimension of open sets is s.

As a consequence of the above property of the family of Euclidean balls in $\mathbf{R}^n$, we have that it is a uniform family of spaces of homogeneous type.

It was first observed by Calderón and Torchinsky [1] that this is not the general case, even when the balls are convex sets in $\mathbf{R}^n$.

Let us consider the two dimensional case $n = 2$. Let $\lambda \geq 1$ be given and set $A_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. The generalized dilations induced on $\mathbf{R}^2$ by A_λ are given by $T_t^\lambda = \begin{pmatrix} t^\lambda & 0 \\ 0 & t \end{pmatrix}$, $t > 0$. If $\|\cdot\|$ is a norm in $\mathbf{R}^2$ then the equation $\|T_t^\lambda x\| = 1$, for $x \in \mathbf{R}^2 - \{0\}$ and λ given has only one solution $t(x) = t_\lambda(x)$. The usual case of parabolic spaces arise when $\lambda = 2$ and the norm is the Euclidean one.

The distance from x to 0 is given by $\frac{1}{t(x)}$. In other words $\rho(x) = \frac{1}{t(x)}$ if $x \neq 0$ and $\rho(0) = 0$. Notice that ρ depends on λ and on the particular norm $\|\cdot\|$ chosen.

Let us consider the case $\|x\| = \|x\|_1 = |x_1| + |x_2|$. Since $B_\rho(0, r) = T_r(B_\rho(0, 1)) = T_r(B_{\|\cdot\|}(0, 1))$ we see that $B_\rho(0, r)$ is the rhombus centered at the origin with vertices at the points:

$$T_r((1, 0)) = (r^\lambda, 0)\,; T_r((-1, 0)) = (-r^\lambda, 0);$$

$$T_r((0, 1)) = (0, r)\,\text{and}\,T_r((0, -1)) = (0, -r).$$

For r large consider the ball $B_\rho(v, 1) = B_{\|\|_1}(v, 1)$ centered at the vertex $v = (r^\lambda, 0)$ of $B_\rho(0, r)$. It is easy to see that the intersection of $B_\rho(0, r)$ and $B_\rho(v, 1)$

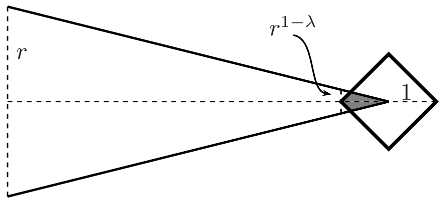

Fig. 1. The situation for $\rho^{1,\lambda}$

contains ρ balls of radii at most $r^{1-\lambda}$. Hence with $\lambda > 1$ there is no chance for the above observed property of the euclidean balls. Let us denote by $\rho^{1,\lambda}$ this metric.

On the other hand, if instead of $\|x\|_1$ we use $\|x\|_\infty = \sup\{|x_1|, |x_2|\}$ we recover for $\rho^{\infty,\lambda}$ that property of Euclidean balls. In fact, the $\rho^{\infty,\lambda}$ balls are now rectangles of the form $[-r^\lambda, r^\lambda] \times [r, r]$ and at no point of the boundary such small angles occur. Moreover $\rho^{1,\lambda}$ and $\rho^{\infty,\lambda}$ are equivalent. So that the property we are looking for is not invariant by equivalence of quasi-distances.

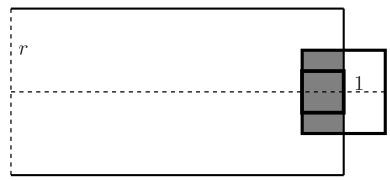

Fig. 2. The situation for $\rho^{\infty,\lambda}$

The issue ♯ 32 of the preprint series "Trabajos de Matemática" published in 1981 by the IAM (Buenos Aires), contains the preprint "A well-behaved quasi-distance for spaces of homogeneous type" by Roberto Macías and Carlos Segovia. There the authors show that it is always possible to find an equivalent quasi-distance on a given space of homogeneous type such that balls are spaces of homogeneous type.

The construction of Macías and Segovia is based on an iterative process of composition of the neighborhoods of the diagonal of $X \times X$.

In this paper we aim to explore the relationship between quasi-distances on X and properties of families of neighborhoods of the diagonal in $X \times X$. We apply the construction of Macías and Segovia to show a somehow stronger version of the uniform regularity of δ-balls for an adequate δ.

2 Quasi-distance on X and diagonal neighborhoods in $X \times X$

Let (X, d) be a quasi-metric space and let K be the triangular constant for d. In other words, d is a nonnegative symmetric function on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$ and $d(x, z) \leq K(d(x, y) + d(y, z))$ for every $x, y, z \in X$. We may think that the diagonal neighborhoods induced by d are given as a function $V : \mathbf{R}^+ \to \mathcal{P}(X \times X)$, where $\mathbf{R}^+$ is the set of positive real numbers and $\mathcal{P}(X \times X)$ is the set of all subsets of $X \times X$. In fact, define

$$V(r) = \{(x, y) \in X \times X : d(x, y) < r\}.$$

It is easy to prove from the basic properties of d that this family of sets satisfies the following properties

(a) $V(r_1) \subseteq V(r_2)$ if $r_1 \leq r_2$;
(b) $\cup_{r>0} V(r) = X \times X$;
(c) $\cap_{r>0} V(r) = \Delta$, the diagonal of $X \times X$;
(d) there exists a constant $\mathcal{K}(= 2K)$ such that $V(r) \circ V(r) \subset V(\mathcal{K}r)$ for every $r > 0$. Here $A \circ B$ is the composition $\{(x,z) \in X \times X : \exists y \in X$ such that $(x,y) \in B$ and $(y,z) \in A\}$ of A and B;
(e) each $V(r)$ is symmetric, i.e. $V(r) = V^{-1}(r)$.

The main result of this section is the converse of the above remark.

Theorem 1. *Let X be a set and let $V : \mathbf{R}^+ \to \mathcal{P}(X \times X)$ be a given family of subsets of $X \times X$ satisfying properties (a), (b), (c), (d) and (e) above. Then there exists a quasi-distance d on X such that for every $0 < \gamma < 1$ we have that $V(\gamma r) \subseteq V_d(r) \subseteq V(r)$ for every $r > 0$, where $V_d(r) = \{(x,y) : d(x,y) < r\}$.*

Proof. Given x and y two points in X, define

$$d(x,y) = \inf\{r : (x,y) \in V(r)\}$$

From property (b) of the function V we see that d is well defined as a function from $X \times X$ to the set of nonnegative real numbers.

From (c) a couple of the form (x,x) belongs to every $V(r)$, hence $d(x,x) = 0$. On the other hand, (c) also implies that if $x \neq y$ then for some $r > 0$ the pair (x,y) does not belong to $V(r)$. Now, from (a), (x,y) does not belong to any $V(s)$ with $s < r$. Hence $d(x,y) > 0$. The symmetry of d follows from the symmetry of each $V(r)$. Let us check the triangular inequality. For x, y, z in X and $\varepsilon > 0$ there exist r_1 and r_2 such that

$$r_1 < d(x,y) + \varepsilon \qquad \text{and} \qquad (x,y) \in V(r_1);$$

$$r_2 < d(y,z) + \varepsilon \qquad \text{and} \qquad (y,z) \in V(r_2);$$

Then $(x,z) \in V(r_2) \circ V(r_1)$ which from property (a) of the family V is contained in $V(r_1 + r_2) \circ V(r_1 + r_2)$. Now, from (d) we have that $V(r_1 + r_2) \circ V(r_1 + r_2) \subset V(\mathcal{K}(r_1 + r_2))$. Hence (x,z) belongs to $V(\mathcal{K}(r_1 + r_2))$, so that, from the very definition of d we have that

$$d(x,z) \leq \mathcal{K}(r_1 + r_2) < \mathcal{K}(d(x,y) + d(y,z)) + 2\mathcal{K}\varepsilon.$$

Which proves the triangle inequality for d with $K = \mathcal{K}$ and d is a quasi-distance on X.

Let us next show that the two families of neighborhoods of Δ, V and V_d are equivalent. Take any $0 < \gamma < 1$. Assume that $(x,y) \in V(\gamma r)$, then $d(x,y) \leq \gamma r < r$ and $(x,y) \in V_d(r)$. On the other hand, if $(x,y) \in V_d(r)$, then $d(x,y) < r$ so that, for some $s < r$, $(x,y) \in V(s) \subset V(r)$.

We say, as usual, that two quasi-distances d and δ defined on X are equivalent if the function $\frac{d}{\delta}$ is bounded above and below by positive constants on $X \times X - \Delta$. Precisely $d \sim \delta$ if there exist two constants $0 < c_1 \leq c_2 < \infty$ such that $c_1\delta(x,y) \leq d(x,y) \leq c_2\delta(x,y)$ for every $x, y \in X$. Given V and W two families of neighborhoods of the diagonal, defined as before, we say that V and W are equivalent and we write $V \approx W$ if there exist constants $0 < \gamma_1 \leq \gamma_2 < \infty$ such that

$$V(\gamma_1 r) \subseteq W(r) \subseteq V(\gamma_2 r),$$

for every $r > 0$.

Given a quasi-distance d, we can assign to d a family V_d of neighborhoods of Δ satisfying (a) to (e) in the standard way, $V_d(r) = \{(x,y) : d(x,y) < r\}$.

On the other hand the construction given in the above theorem provides a method to assign to every family V of neighborhoods of Δ, satisfying (a) to (e), a quasi distance d_V such that $V \sim V_{d_V}$.

The next proposition contain basic properties of these equivalences. Let us denote by $\mathcal{V}$ the class of all families satisfying (a) to (e) and by $\mathcal{D}$ the set of all quasi distances in X.

Proposition 1.
(i) For every $V \in \mathcal{V}$ we have that $V \approx V_{d_V}$;
(ii) for every $d \in \mathcal{D}$ we have that $d \sim d_{V_d}$;
(iii) for V and W in $\mathcal{V}$ we have that $V \approx W$ if and only if $d_V \sim d_W$;
(iv) for d and δ in $\mathcal{D}$ we have that $d \sim \delta$ if and only if $V_d \approx V_\delta$.

The above proposition shows that we can identify $\mathcal{D}/\sim$ and $\mathcal{V}/\approx$ through

$$\begin{array}{ccc} \mathcal{D} & \overset{V_d}{\underset{d_V}{\rightleftarrows}} & \mathcal{V} \\ \Pi_{\mathcal{D}} \downarrow & & \downarrow \Pi_{\mathcal{V}} \\ \mathcal{D}/\sim & \overset{H}{\longleftrightarrow} & \mathcal{V}/\approx \end{array}$$

where $H(\overline{d}) = \overline{\overline{V_d}}$ for any $d \in \overline{d}$. Here $\overline{d}$ denotes the $\sim$ class of $d \in \mathcal{D}$ and $\overline{\overline{V_d}}$ denotes the $\approx$ class of $V \in \mathcal{V}$.

Several problems in generalized harmonic analysis are invariant under changes of equivalent quasi-distances. For example the Hölder-Lipschitz as BMO type spaces are the same for d and δ if $d \sim \delta$. The Muckenhoupt classes do not change by changing equivalent quasi-metrics, etcetera. Even some kernels like fractional integrals define equivalent operators. When a particular property of the quasi-distance function becomes relevant for a specific problem in harmonic analysis, the question is whether or not such properties holds for at least one representative of each class in $\mathcal{D}/\sim$.

For example the well known result of R. Macías and C. Segovia [2] asserting that each quasi-distance is equivalent to a power of a distance can be written saying that the mapping

$$D \times \mathbf{R}^+ \overset{J}{\longrightarrow} \mathcal{D}/\sim$$

given by

$$(\rho, \alpha) \longrightarrow J(\rho, \alpha) = \overline{\rho^\alpha}$$

is onto, where D is the family of all distances in X. Notice that J can not be one to one since $\rho^\alpha = (\rho^{1/2})^{2\alpha}$ and $\rho^{1/2}$ is still a distance if ρ is.

We are interested in finding an as large as possible family of measurable subsets Y of X such that (Y, d_Y, μ_Y) is a space of homogeneous type, where d_Y is the restriction of d to Y and μ_Y is the restriction of μ to the measurable subsets of Y. Moreover, from the point of view of its applications in problems of harmonic analysis, we would like to have large families $\mathcal{F}$ of sets Y such that the spaces (Y, d_Y, μ_Y) are uniformly spaces of homogeneous type. If $\mathcal{F}$ is such a family of subsets of X we briefly say that $\mathcal{F}$ has the u.s.h.t. property. Of course the u.s.h.t. property depends of the distance d and of the measure μ. But it is easy to check that if $\mathcal{F}$ satisfies the u.s.h.t. property with respect to (X, d, μ) and $\delta \sim d$, then $\mathcal{F}$ satisfies the u.s.h.t. property with respect to (X, δ, μ). We shall actually deal with a somehow improved version of u.s.h.t.

3 Regularization of neighborhoods of Δ

The smoothing procedure designed by Macías and Segovia in [2] is based in a self-similarity argument which we proceed to describe. Let d be a quasi-distance on X with constant K. For simplicity of notation, set $U(r) = V_d(r)$. In other words $U(r) = \{(x, y) : d(x, y) < r\}; r > 0$.

Pick a fixed positive number α less than $\frac{1}{2K}$. We start by the construction of a sequence $U(r, n)$ in the following way

$U(r, 0) = U(r)$

$U(r, 1) = U(ar) \circ U(r) \circ U(ar)$

$U(r, n) = U(a^n r) \circ U(r, n-1) \circ U(a^n r)$.

Lemma 1. *For every $r > 0$ and every n we have that*

$$U(r) \subset U(r, n) \subset U(3k^2 r).$$

Proof. Since each $U(s)$ contains the diagonal Δ of $X \times X$, then it is clear that $U(r) \subset U(r, n)$ for $n = 0, 1, 2, \ldots$. So that, we only have to prove that $U(r, n) \subset U(3k^2, r)$. Let $(x, y) \in U(r, n)$. Then, there exists a finite sequence $x = x_0, x_1, x_2, \ldots, x_n, y_n, \ldots, y_2, y_1, y_0 = y$ of points in X such that

$(x_n, y_n) \ \in U(r) = U(r, 0)$,
$(x_j, x_{j+1}) \in U(a^{n-j}r)$, and
$(y_j, y_{j+1}) \in U(a^{n-j}r)$, $j = 0, 1, \ldots, n$.

Let us now estimate $d(x, y)$ by repeated use of the triangle inequality,

$$
\begin{aligned}
d(x,y) = d(x_0,y_0) &\le K^2[d(x_0,x_n) + d(x_n,y_n) + d(y_n,y_0)] \\
&\le K^2\left[\textstyle\sum_{i=0}^{n-1} d(x_i,x_{i+1})K^{n-i} + d(x_n,y_n) + \sum_{i=0}^{n-1} d(y_i,y_{i+1})K^{n-i}\right] \\
&< K^2\left[\textstyle\sum_{i=0}^{n-1} a^{n-i}rK^{n-i} + r + \sum_{i=0}^{n-1} a^{n-i}rK^{n-i}\right] \\
&= rK^2\left[1 + 2\textstyle\sum_{i=0}^{n-1}(aK)^{n-i}\right] \\
&< 3K^2 r.
\end{aligned}
$$

The inequality $d(d, y) < 3K^2r$ proves that $(x, y) \in U(3K^2r)$, and the lemma is proved.

Given a positive r, define

$$V(r) = \bigcup_{n=0}^{\infty} U(r, n)$$

Lemma 2. *The family $V(r)$ is equivalent to the family $U(r)$ and satisfies properties (a) to (e) in Section 1.*

Proof. Let us start by noticing that from the above lemma, for every $r > 0$ we have $U(r) \subset V(r) \subset U(3k^2r)$. If $r_1 \le r_2$, then $U(a^j r_1) \subseteq U(a^j r_2)$. Hence $U(r_1, n) \subseteq U(r_2, n)$ and $V(r_1) \subseteq V(r_2)$, which proves (a). Properties (b) and (c) for $V(r)$ follow from properties (b) and (c) for $U(r)$ and from the above lemma. The symmetric of $V(r)$ follows from the symmetric of $U(r)$. Let us finally check that $V(r)$ satisfies property (d). In fact from Lemma 1 we have that

$$V(r) \circ V(r) \subset U(3K^2r) \circ U(3K^2r) \subset U(6K^3r) \subset V(bK^3r),$$

is (d) for $V(r)$ with $\mathcal{K} = 6K^3$.

Let $\delta = d_V$. Then $\delta \sim d$. The basic scheme of the procedure is this

$$d \to U = V_d \to V \to d_V = \delta.$$

Here the middle arrow is the regularization procedure of Macías and Segovia and the other two are the canonical mappings.

Since the regularization procedure produce equivalent neighborhood systems we obtain a new quasi-distance δ which is equivalent to d.

4 The main result

In this section we aim to prove the following result.

Theorem 2. *Let (X,d) be a quasi-metric space. Then there exist a quasi-distance δ on X and a positive constant γ such that*

(i) $\delta \sim d$;
(ii) the δ-balls are open sets;
(iii) for every $x \in X$, every choice of r and R with $0 < r < R$ and every $x \in B_\delta(x_0, R)$ there exists a point $\xi \in X$ such that

$$B(\xi, \gamma_r) \subset B_\delta(x, R) \cap B(y, r).$$

We are using the notation B_δ for the δ-balls and B for the balls defined by the original quasi-distance d on X. We may think of B as the testing balls for B_δ, the tested set. Hence the family of d-balls can be changed to the family of balls corresponding to any other quasi-distance equivalent to d. In particular by the δ-balls.

Let us first notice that it is enough to prove the theorem for the case of a distance ρ on X instead of a general quasi-distance d. In fact, given d on X there exists ρ a distance on X and $\alpha \geq 1$ such that $d \sim \rho^\alpha$. Let us assume that we know the result for ρ. In other words, there exist a quasi-distance δ_ρ and $\gamma_\rho > 0$ with $\delta_\rho \sim \rho$, the δ_ρ-balls open sets and, for every $x \in X$ every $0 < s < S$ and every $y \in B_{\delta_\rho}(x, S)$ we have that for some $\xi \in X$, that

$$B_\rho(\xi, \gamma_\rho s) \subset B_{\delta_\rho}(x, S) \cap B_\rho(y, s).$$

Take $\delta = \delta_\rho^\alpha \sim \rho^\alpha \sim d$. Since $B_\delta(z, r) = B_{\delta_\rho}(z, r^{1/\alpha})$ we see that δ-balls are open sets. Notice also that $B_{\rho^\alpha}(z, r) = B_\rho(Z, r^{1/\alpha})$. Take now $x \in X, 0 < r < R$ and $y \in B_\delta(x, R)$. Then, with $s = r^{1/\alpha}$, $S = R^{1/\alpha}$ we have that $y \in B_{\delta_\rho}(x, S)$, hence from our assumption, we get

$$\begin{aligned} B(\xi, c\gamma_\rho^\alpha r) &\subset B_{\rho^\alpha}(\xi, \gamma_\rho^\alpha r) = B_\rho(\xi, \gamma_\rho s) \\ &\subseteq B_{\delta_\rho}(x, S) \bigcap B_\rho(y, s) \subseteq B_\delta(x, R) \bigcap B(y, \bar{c}r). \end{aligned}$$

From now on we shall assume that d is a distance on X, i.e. $K = 1$. One more reduction of Theorem ()is in order. Assume that we can prove that there exists a positive λ such that for every distance d on X there exists a quasi-distance δ on X satisfying (i), (ii) and (iii) with $R = 1$. Then the theorem follows. In fact, if d is a distance so is μd for $\mu > 0$. Since λ does not depend on the given distance, the scaling argument is clear. To obtain our main result we only have to prove the following statement.

Proposition 2. *There exists a positive constant λ such that, for every metric metric space (X,d), there exists a quasi-distance δ on X, equivalent to d, such* *that the δ-balls are open sets and for every $x \in X, 0 < r < 1$ and $y \in B_\delta(x,1)$ there exists $\xi \in X$ such that*

$$B(\xi, \lambda r) \subset B_\delta(x,1) \cap B(y,r).$$

For a metric space the result of Lemma 1 reads

$$U(r) \subset U(r,n) \subset U(3r),$$

for every $r > 0$ and every $n \in \mathbf{N}$ if $U(r) = \{(x,y) : d(x,y) < r\}$. For fixed $n \in \mathbf{N}$ the family $\mathcal{U}_n = \{U(r,n) : r > 0\}$ satisfies properties (a) to (e). Then

$$\delta_n(x,y) = \inf\{r > 0 : (x,y) \in U(r,n)\}$$

is a quasi-distance on X, equivalent to d. Moreover, since $U(r) \subset U(r,n) \subset U(3r)$ for every n, the equivalence constants are independent of n. Also the triangular constants are uniformly bounded above. In fact, if $x, y, z \in X$ and $\varepsilon > 0$, pick r_1 and r_2 such that $r_1 < \delta_n(x,y) + \varepsilon$ and $r_2 < \delta_n(y,z) + \varepsilon$ with $(x,y) \in U(r_1,n)$ and $(y,z) \in U(r_2,n)$. Thus

$$\begin{aligned}(x,y) \in U(r_1,n) \circ U(r_2,n) &\subseteq U(r_1+r_2,n) \circ U(r_1+r_2,n)\\ &\subseteq U(3(r_1+r_2)) \circ U(3(r_1+r_2))\\ &\subseteq U(6(r_1+r_2)),\end{aligned}$$

which proves that $\delta_n(x,z) \leq 6(\delta_n(x,y) + \delta_n(y,z))$. Since $U(r,n) \subset V(r)$ for every $r > 0$ and every $n \in \mathbf{N}$ we have that $\delta_n(x,y) \geq \delta(x,y)$. On the other hand, since $V(r) = \bigcup_n U(r,n)$ we have the pointwise convergence of $\delta_n(x,y)$ to $\delta(x,y)$.

Lemma 3. *For every $x \in X$ and every $r > 0$ we have that*

$$B_{\delta_n}(x,r) = \{y : (x,y) \in U(r,n)\}$$

and that

$$B_\delta(x,r) = \{y : (x,y) \in V(r)\} = \bigcup_n B_{\delta_n}(x,r).$$

Proof. Applying the argument used in Section 3 for the case of a distance d, we can take the number a to be $\frac{1}{4}$. Hence $U(r,0) = U(r) = \{(x,y) : d(x,y) < r\}$ and $U(r,n) = U(\frac{r}{4^n}) \circ U(r,n-1) \circ U(\frac{r}{4^n})$ for $n \in \mathbf{N}$.

If $y \in B_{\delta_n}(x,r)$, then $\delta_n(x,y) < r$, so that $(x,y) \in U(s,n)$ for some $s < r$ and $(x,y) \in U(r,n)$. Take now $(x,y) \in U(r,n)$, then there exists a chain of points in X

$$x_0 = x, x_1, x_2, \cdots, x_{n-1}, x_n; y_n, y_{n-1}, \cdots, y_2, y_1, y_0 = y$$

such that $d(x_n, y_n) < r$; $d(x_{n-j+1}, x_{n-j}) < r4^{-j}$; and $d(y_{n-j+1}, y_{n-j}) < r4^{-j}$ for $j = 0, 1, \cdots, n$.

Since we have a finite number of strict inequalities we certainly have that for some $s < r$ we must also have that $d(x_n, y_n) < s$; $d(x_{n-j+1}, x_{n-j}) < s4^{-j}$; and $d(y_{n-j+1}, y_{n-j}) < s4^{-j}$; for $j = 0, 1, 2, \cdots, n$. This fact proves that $(x, y) \in U(s, n)$ for some $s < r$, then $\delta_n(x, y) \leq s < r$. Hence $y \in B_{\delta_n}(x, r)$.

Notice also that since d is a distance the sets $U(r)$ and $U(r, n)$ are open sets in $X \times X$. The same is true for each $V(r)$. Hence their sections B_{δ_n} and B_δ are open sets in X.

The proof of Proposition 2 is then reduced to the proof of the next result

Proposition 3. *Let (X, d) be a metric space. Then, for every $n \in \mathbf{N}$, for every $i \in \{1, \cdots, n\}$ for every $x \in X$ and for every $y \in B_{\delta_n}(x, 1)$, there exists a point $\xi \in X$ such that*

$$B(\xi, 4^{-i-1}) \subseteq B_{\delta_n}(x, 1) \cap B(y, 4^{-i}).$$

Also $B(y, 4^{-i}) \subseteq B(\xi, 4^{-i+1})$.

Proof. Since $y \in B_{\delta_n}(x, 1)$, then $\delta_n(x, y) < 1$. From the definition of δ_n we know that there exists $s \in (0, 1)$ such that $(x, y) \in U(s, n)$. In other words, there exists $s \in (0, 1)$ and a finite sequence in X

$$C : x = x_0, x_1, x_2, \cdots, x_{n-1}, x_n; y_n, y_{n-1}, \cdots, y_2, y_1, y_0 = y$$

such that $d(x_n, y_n) < s$, $d(x_{n-j+1}, x_{n-j}) < s4^{-j}$ and $d(y_{n-j+1}, y_{n-j}) < s4^{-j}$ for $j = 0, 1, 2, \cdots, n$.

Fix now i in $\{1, \cdots, n\}$. The proposition is valid with $\xi = y_{n-i}$ if we prove

(A) $B(y_{n-i}, 4^{-i-1}) \subseteq B_{\delta_n}(x, 1)$;

(B) $B(y, 4^{-i}) \subseteq B(y_{n-i}, 4^{-i+1})$;

(C) $B(y_{n-i}, 4^{-i-1}) \subseteq B(y, 4^{-i})$.

Proof of (A): Take $z \in B(y_{n-i}, 4^{-i-1})$. From Lemma 3, in order to show that $z \in B_{\delta_n}(x, 1)$ we only need to check that (z, x), or (x, z), is an element of $U(1, n)$. Let us use some points in the chain C to produce another chain C' joining x to z,

$$C' : x = x_0, x_1, x_2, \cdots, x_{n-1}, x_n; y_n, y_{n-1}, \cdots, y_{n-i}, \underbrace{z, z, \cdots, z}_{n-i}$$

Let us see that C' is an admissible chain to show that $(x, z) \in U(1, n)$,

$d(x_n, y_n) < s,$
$d(x_{n-j+1}, x_{n-j}) < s4^{-j},\ j = 0, \cdots, n$
$d(y_{n-j+1}, y_{n-j}) < s4^{-j},\ j = 0, \cdots, i$
$d(y_{n-i}, z) < 4^{-i-1}$, since z$\in B(y_{n-i}, 4^{-i-1})$
$d(z, z) = 0.$

Since $s < 1$ we have that C' satisfies the required condition for $(x, z) \in U(1, n)$.

Proof of (B): Take $z \in B(y, 4^{-i})$. Then

$$
\begin{aligned}
d(z, y_{n-i}) &\leq d(z, 0) + d(y, y_{n-i}) \\
&< 4^{-i} + d(y_0, y_{n-i}) \\
&\leq 4^{-i} + \textstyle\sum_{j=i+1}^{n} d(y_{n-j+1}, y_{n-j}) \\
&< 4^{-i} + s \textstyle\sum_{j=i+1}^{n} 4^{-j} < 4^{-i+1}.
\end{aligned}
$$

Proof of (C):Take $z \in B(y_{n-i}, 4^{-i-1})$, then

$$
\begin{aligned}
d(z, y) &\leq d(z, y_{n-i}) + d(y_{n-i}, y) \\
&< 4^{-i-1} + \tfrac{s}{3} 4^{-i} < 4^{-i}
\end{aligned}
$$

Let us finally point out that, even when the preceding proof is essentially the given in [2], Theorem 2 is giving more information than doubling. If for example (X, d, μ) is a s-normal space then the family of δ-balls is a uniform family of s-normal subspaces.

References

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[2] R. A. Macías and C. A. Segovia: *A well behaved quasi-distance for spaces of homogeneous type*, Trabajos de Matemática, IAM, 32 (1981), 1-18.