DISCRETE APPROXIMATION OF SPACES OF HOMOGENEOUS TYPE

AIMAR HUGO, CARENA MARILINA, IAFFEI BIBIANA

ABSTRACT. In this note we combine the dyadic families introduced by M. Christ in [4] and the discrete partitions introduced by J. M. Wu in [12] in order to get approximation of a compact space of homogeneous type by a uniform sequence of finite spaces of homogeneous type. The convergence holds in the sense of a metric built on the Hausdorff distance between sets and on the Kantorovich-Rubinshtein metric between measures.

INTRODUCTION

In order to introduce the problem considered in this paper, let us start by two very classical examples.

If for each $n \in \mathbb{N}$ we define on the Borel sets of the real numbers the normalize counting measure supported on $S_n = \{i/n : i = 0, 1, 2, \dots, n\}$, given by $\mu_n(A) = \frac{1}{n+1} \operatorname{card}(\{i : 0 \le i \le n \text{ and } i/n \in A\})$, we have that

$$S_n \xrightarrow{d_H} [0,1] \qquad \text{and} \qquad \mu_n \xrightarrow{w*} m,$$

where the d_H-convergence is the Hausdorff convergence of compact sets, the $w*$-convergence is the weak star converge of measures, and m is the Lebesgue measure on the closed interval $[0,1]$. In other words, perhaps the most elementary probabilistic space of homogeneous type $([0,1], |\cdot|, m)$ can be approximated in the Hausdorff-Kantorovich sense by a sequence of finite spaces. Moreover, the spaces $(S_n, |\cdot|, \mu_n)$ are themselves spaces of homogeneous type with doubling constants bounded above by a fixed number independent of $n \in \mathbb{N}$. In fact given $n \in \mathbb{N}$, $x \in S_n$ and $0 < r \le 1$, choosing $j \in \mathbb{Z}$ such that $j/n < r \le (j+1)/n$ we have

$$\begin{aligned}(n+1)\mu_n(B(x,2r)) &\le 2(2j+1)+1 \\ &< 4(j+1) \\ &\le 4(n+1)\mu_n(B(x,r)),\end{aligned}$$

where $B(x,r) = \{y : |x-y| < r\}$.

Most interesting is the case of the Cantor set C. Let F be the Cantor function extended to $\mathbb{R}$ as a continuous function by defining $F(x) = 1$ for $x \ge 1$ and $F(x) = 0$ for $x \le 0$. Let μ the unique probabilistic Borel measure on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for every $a < b$. It is well know (see [10] and [13]), realizing the Cantor set as the attractor of an iterated function system, that $(C, |\cdot|, \mu)$ is a space of

Key words and phrases. quasi-metric space, doubling measure, Hausdorff-Kantorovich metric.

homogeneous type en even normal. Let us write

$$C=\bigcap_{n=1}^{\infty} C_n, \quad C_n=\bigcup_{j=1}^{2^n} I_n^j, \quad I_n^j=[a_n^j, b_n^j]\,,$$

where C_n is the set obtained as the n-th step in the construction of the Cantor set. For each $n\in\mathbb{N}$, let $L_n=\{a_n^j : j=1,2,\ldots,2^n\}$, in other words, L_n is the collection of all the left points of each interval in C_n. Notice that for each n we have that $d_H(L_n, C)\leq 2/3^n$. Then $L_n \xrightarrow{d_H} C$ when $n\to\infty$. Let μ_n be the discrete measure defined on L_n by $\mu_n(\{x\})=2^{-n}$ for each $x\in L_n$. Then $\mu_n \xrightarrow{w*} \mu$. In fact, for $\varphi\in\mathcal{C}([0,1])$ we have

$$\int_{[0,1]} \varphi(x)\, d\mu_n(x)=\frac{1}{2^n}\sum_{j=1}^{2^n}\varphi(a_n^j)\,.$$

On the other hand, for fixed n,the partition of $[0,1]$ given by

$$P_n=\{x_\ell=\ell/3^n : \ell=0,1,2,\ldots,3^n\}$$

contains L_n. From the construction of F as a limit of the continuous and piecewise linear functions F_k, one easily see that $F_k(x_\ell)=F_n(x_\ell)$ for every $\ell=0,1,\ldots,3^n$ and every $k\geq n$. Then $F(x_\ell)=F_n(x_\ell)$ and every $\ell=0,1,\ldots,3^n$, so that

$$\begin{aligned}\sum_{\ell=0}^{3^n-1}\varphi(x_\ell)[F(x_{\ell+1})-F(x_\ell)] &= \sum_{\ell=0}^{3^n-1}\varphi(x_\ell)[F_n(x_{\ell+1})-F_n(x_\ell)]\\ &= \frac{1}{2^n}\sum_{j=1}^{2^n}\varphi(a_n^j)\,.\end{aligned}$$

The last inequality follows from the fact that $F_n(x_{\ell+1})-F_n(x_\ell)=2^{-n}$ if $x_\ell\in L_n$ and it vanishes elsewhere. Hence

$$\int_{[0,1]}\varphi(x)\,d\mu_n(x)=\sum_{\ell=0}^{3^n-1}\varphi(x_\ell)[F(x_{\ell+1})-F(x_\ell)] \xrightarrow[n\to\infty]{} \int_{[0,1]}\varphi(x)\,dF(x)\,,$$

so that $\mu_n \xrightarrow{w*} \mu$.

Let us next prove that there exist $A\geq 1$ such that $(L_n, |\cdot|, \mu_n)$ is space of homogeneous type with doubling constant bounded by A, for every n. Let us notice that L_n can be obtained by dividing by 3^n all the non-negative integers whose expansion in basis 3 do not contain the digit 1 and having at most n digits. So that each point $x\in L_n$ can be identified with an n-tuple $(x_1,x_2,\ldots,x_n)$ where each x_i is zero or two. With this notation, following [2], define $d_n: L_n\times L_n\to\mathbb{R}_0^+$ by

$$d_n(x,y)=\begin{cases}0, & \text{if } x=y,\\ 3^{-j}, & \text{if } x_i=y_i \text{ for every } i<j \text{ and } x_j\neq y_j.\end{cases}$$

It is easy to see that d_n is a distance on L_n. Let us first show that (L_n, d_n, μ_n) is a uniform family of spaces of homogeneous type, in the sense that there exists a constant A such that the inequalities

$$0<\mu_n\left(B_{d_n}(x,2r)\right)\leq A\mu_n\left(B_{d_n}(x,r)\right)<\infty \tag{1}$$

hold for each $x \in L_n$, $r > 0$ and $n \in \mathbb{N}$, where $B_{d_n}(x,r) = \{y \in L_n : d_n(x,y) < r\}$. Notice that for $x \in L_n$ and $j \in \mathbb{N}$ we have

$$B_{d_n}\left(x, 3^{-j}\right) = \{y \in L_n : y_i = x_i,\, i = 1, 2, \dots, j\} \ ,$$

hence

$$\operatorname{card}\left(B_{d_n}\left(x, 3^{-j}\right)\right) = \begin{cases} 2^{n-j}, & j \leq n, \\ 1, & j \geq n. \end{cases}$$

So that

$$\mu_n\left(B_{d_n}\left(x, 3^{-j}\right)\right) = \begin{cases} 2^{-j}, & j \leq n, \\ 2^{-n}, & j \geq n. \end{cases}$$

From this estimate, for a given $0 < r < 1$, choosing $j \in \mathbb{N}$ such that $3^{-j} < r \leq 3^{1-j}$ we have (1) with $A = 4$. Observe that given $n \in \mathbb{N}$ and $x, y \in L_n$, $x \neq y$, with $d_n(x,y) = 3^{-j}$, we necessarily have that

$$x - y = \sum_{i=j}^{n} 3^{-i}(x_i - y_i),$$

from which we obtain the inequalities

$$d_n(x,y) \leq |x - y| \leq 3 d_n(x,y),$$

for every $n \in \mathbb{N}$ and every $x, y \in L_n$. Hence also $(L_n, |\cdot|, \mu_n)$ is a uniform sequence of spaces of homogeneous type. In fact, for $n \in \mathbb{N}$, $x \in L_n$ and $r > 0$ we have

$$\begin{aligned} \mu_n\left(B(x, 2r)\right) &\leq \mu_n\left(B_{d_n}(x, 2r)\right) \\ &\leq 4^3 \mu_n\left(B_{d_n}(x, r/3)\right) \\ &\leq 4^3 \mu_n\left(B(x, r)\right). \end{aligned}$$

The aim of this paper is to show that the situation of the above examples is general. More precisely, we shall prove that each probabilistic compact space of homogeneous type can be approximated in the Hausdorff-Kantorovich sense by a sequence of finite spaces of homogeneous type with a uniform bound for the doubling constant.

To prove our result we use the techniques introduced by J. M. Wu in [12] to produce partitions on the discrete approximation, and those introduced by M. Christ in [4] to built dyadic type families on spaces of homogeneous type.

In Section 1 we introduce the Hausdorff-Kantorovich distance, and in Section 2 we prove a completeness type property for the families of spaces of homogeneous type with bounded doubling constant. The main result, providing the discrete approximation of a given space of homogeneous type, is contained in Section 3.

1. The Hausdorff-Kantorovich quasi-metric

Let X be a given set. A function $\rho : X \times X \to \mathbb{R}^+ \cup \{0\}$ is called a *quasi-distance* if ρ is symmetric, ρ vanishes on the diagonal of $X \times X$, ρ is faithful, i.e. $\rho(x,y) = 0$ implies $x = y$, and there exists a constant $\Lambda \geq 1$ such that the inequality $\rho(x,y) \leq \Lambda(\rho(x,z) + \rho(z,y))$ holds for every $x, y, z \in X$. The family $\mathcal{N}_x$ of subsets E of X for which, for some $r > 0$, $B_\rho(x,r) := \{y \in X : \rho(x,y) < r\} \subseteq E$ is a neighborhood system for a topology τ on X. The sets $B(x,r)$ are called the ρ-balls or simply the balls in X. The basic result concerning quasi-metric spaces is a theorem due to Macías and Segovia [9] which actually proves that for each quasi-distance ρ on X there exist a distance d on X and a number $\xi \geq 1$ depending only on Λ such that $\rho \simeq d^{\xi}$. In other words, there exist constants c_1 and c_2 which depend only on Λ such that the inequalities

$$c_1\rho(x,y) \leq d^{\xi}(x,y) \leq c_2\rho(x,y), \tag{2}$$

hold for every $x, y \in X$. In particular the topology τ introduced through the neighborhood system $\mathcal{N}_x$ given by the ρ-balls, is the metric topology induced on X by d. Hence each topologic concept introduced further can be regarded as a metric one.

Throughout this paper (X,ρ) shall be a compact quasi-metric space. With d we shall always denote a distance for which there exist ξ, c_1 and c_2 constants for which (2) holds. For any closed subset Y of X, the quasi-metric space (Y,ρ) is a compact subspace of (X,ρ).

To accomplish our aims we start by introducing a quasi-metric structure on the closed probabilistic subspaces of homogeneous type (Y,ρ,μ) of (X,ρ) with uniform upper bounds for the doubling constants. This topology involves the Hausdorff convergence of compact sets and the Kantorovich weak $*$ convergence of probabilities.

Let $\mathcal{K} = \{K \subseteq X : K \neq \emptyset,\ K \text{ compact}\}$. With $[A]_\epsilon$ we shall denote the ϵ-enlargement of the set $A \subset X$; i.e. $[A]_\epsilon = \bigcup_{x\in A} B_\rho(x,\epsilon) = \{y \in X : \rho(y,A) < \epsilon\}$. Here $\rho(x,A) = \inf\{\rho(x,y) : y \in A\}$. Given A and B two sets in $\mathcal{K}$ the Hausdorff quasi-distance from A to B is given by

$$\delta_H(A,B) = \inf\{\epsilon > 0 : A \subseteq [B]_\epsilon \text{ and B} \subseteq [\text{A}]_\epsilon\}.$$

Of course δ_H is the usual Hausdorff distance when ρ itself is a metric. The next result is a corollary of the completeness of the Hausdorff distance (see [7]) and of the above mentioned theorem of Macías and Segovia.

Proposition 1.1. *$(\mathcal{K},\delta_H)$ is a complete quasi-metric space.*

Proof. Set d_H to denote the usual Hausdorff distance on $\mathcal{K}$ associated to d. In other words if $[A]_{\epsilon,d} = \{x \in X : d(x,A) < \epsilon\}$ denotes the ϵ-neighborhood of A with respect to d, then $d_H(A,B) = \inf\{\epsilon > 0 : A \subseteq [B]_{\epsilon,d} \text{ and } B \subseteq [A]_{\epsilon,d}\}$, for $A, B \in \mathcal{K}$. Since for every $\epsilon > 0$ we have that

$$[A]_{(\epsilon/c_1)^{1/\xi},\,d} \subseteq [A]_\epsilon \subseteq [A]_{(c_2\epsilon)^{1/\xi},\,d}$$

we have that

$$\begin{aligned}
\delta_H(A,B) &\geq \inf\{\epsilon > 0 : A \subseteq [B]_{(c_2\epsilon)^{1/\xi},\,d} \text{ and } B \subseteq [A]_{(c_2\epsilon)^{1/\xi},\,d}\} \\
&= \frac{1}{c_2}\left[\inf\{(c_2\epsilon)^{1/\xi} : A \subseteq [B]_{(c_2\epsilon)^{1/\xi},\,d} \text{ and } B \subseteq [A]_{(c_2\epsilon)^{1/\xi},\,d}\}\right]^{\xi} \\
&= \frac{1}{c_2}\, d_H^{\xi}(A,B),
\end{aligned}$$

for every A and B in $\mathcal{K}$. With a similar argument we can show that $\delta_H(A,B) \leq d_H^{\xi}(A,B)/c_1$. Hence $\delta_H \simeq d_H^{\xi}$, with the same constants c_1 and c_2 in (2). Since $(\mathcal{K},d_H)$ is a complete metric space, we have that that δ_H is a quasi distance on $\mathcal{K}$ and $(\mathcal{K},\delta_H)$ is a complete quasi-metric space. $\square$

Let us now introduce the Kantorovich-Rubinshtein distance (known also as the Hutchinson distance, see [1], [8]) on the set of all Borel regular probability measures on the quasi-metric space (X, ρ). Let

$$\mathcal{P}(X) = \{\mu : \mu \text{ is a positive Borel measure on } X \text{ and } \mu(X) = 1\},$$

and let $\mathcal{C}(X)$ be the space of continuous real valued functions on X. Since the Borel σ-algebra induced by the quasi-distance ρ is the same as the one induced by d, we have that every measure μ in $\mathcal{P}(X)$ is regular (see [3]). For $c > 0$, let us denote by $\mathrm{Lip_c}$ the space of all d-Lipschitz continuous functions defined on X with Lipschitz constant equal to one, i.e. $f \in \mathrm{Lip_c}$ if and only if $|f(x) - f(y)| \le cd(x, y)$ for every x and $y \in X$.

Since (X, ρ) is compact, $\delta_K(\mu, \nu) = \sup\left\{\left|\int f\, d\mu - \int f\, d\nu\right| : f \in \mathrm{Lip}_1\right\}$ gives a distance on $\mathcal{P}(X)$ such that the δ_K-convergence of a sequence is equivalent to its weak $*$ convergence to the same limit. Hence, in our situation, the metric space $(\mathcal{P}(X), \delta_K)$ becomes complete.

Even when the results stated in the above paragraph are well known, specially for subsets of the Euclidean space, for the sake of completeness, we shall briefly sketch their proofs.

Let us remind that $\mu_n \xrightarrow{w*} \mu$ if and only if $\int \varphi\, d\mu_n \to \int \varphi\, d\mu$ for every $\varphi \in \mathcal{C}(X)$. Notice that weak star convergence depends only on the topology of X, not on the specific metric or quasi-metric that generates it. Since X is compact, $\mathcal{P}(X)$ is sequentially compact by Prohorov's Theorem (see for example [3]), that is, for every sequence $\{\mu_n\}$ in $\mathcal{P}(X)$ there exist a subsequence $\{\mu_{n_i}\}$ and a measure $\mu \in \mathcal{P}(X)$ such that $\mu_{n_i} \xrightarrow{w*} \mu$. This fact implies that $\mathcal{P}(X)$ is complete with the weak star topology.

Lemma 1.2. *Let $\mu_1, \mu_2 \dots$ and μ be measures in $\mathcal{P}(X)$. Then $\mu_n \xrightarrow{w*} \mu$ if and only if $\delta_K(\mu_n, \mu) \to 0$ when $n \to \infty$.*

The proof follows the lines of Lemma 1.10 in [6], actually the fact that the weak star convergence implies that $\delta_K(\mu_n, \mu) \to 0$ is valid with no changes.

For the converse suppose that $\delta_K(\mu_n, \mu) \to 0$. Notice that since X is compact, then the class $\mathcal{A} = \bigcup_{c>0} \mathrm{Lip_c}$ is a subalgebra of $\mathcal{C}(X)$. Also $\mathcal{A}$ separate points; that is, given two distinct points x and y in X, we can find an f in $\mathcal{A}$ such that $f(x) \neq f(y)$. In fact, given $x, y \in X$ with $x \neq y$, it is enough to take $f(z) = d(x, z)$, which belongs to Lip_1. Since $\mathcal{A}$ contains the constant functions, then from the Stone-Weierstrass theorem for compact metric spaces (see [11]) we have that $\mathcal{A}$ is dense in $\mathcal{C}(X)$. Then given $\varphi \in \mathcal{C}(X)$ and $\epsilon > 0$, there exists $f \in \mathrm{Lip}_c$ for some $c > 0$ such that $|\varphi(x) - f(x)| < \epsilon/3$ for all $x \in X$. Let $n_0 = n_0(\epsilon)$ be such that if $n \ge n_0$ then $\delta_K(\mu_n, \mu) < \epsilon/(3c)$. Then if $n \ge n_0$ we have

$$\begin{aligned} \left|\int \varphi\, d\mu_n - \int \varphi\, d\mu\right| &\le \int |\varphi - f|\, d\mu_n + \left|\int f\, d\mu_n - \int f\, d\mu\right| + \int |\varphi - f|\, d\mu \\ &< 2\epsilon/3 + c\delta_K(\mu_n, \mu) < \epsilon. \end{aligned}$$

Hence $\mu_n \xrightarrow{w*} \mu$.

We are now in position to describe the basic quasi-metric space whose structure and convergence properties are of our interest. Let $\mathscr{X}$ be the set of all couples (Y, μ) such that Y is a closed, and hence compact, subset of X, and μ is a regular Borel probability measure on X. In other words, $\mathscr{X} = \mathcal{K} \times \mathcal{P}$. Given two elements

(Y_i, μ_i) of $\mathscr{X}$, $i = 1, 2$, define

$$\delta\left((Y_1, \mu_1), (Y_2, \mu_2)\right) = \delta_H(Y_1, Y_2) + \delta_K(\mu_1, \mu_2)\,,$$

so that $(\mathscr{X}, \delta)$ becomes a complete quasi-metric space. Let $\mathcal{E}$ be the set of all $(Y, \mu) \in \mathscr{X}$ such that the support of μ is contained in Y, in other words

$$\mathcal{E} = \{(Y, \mu) \in \mathscr{X} : \mathrm{supp}\mu \subseteq Y\},$$

were $\mathrm{supp}\mu$ is the complementary of the largest open set G in X for which $\int \varphi\, d\mu = 0$ for every $\varphi \in \mathcal{C}(X)$ with $\mathrm{supp}\varphi \subseteq G$.

Theorem 1.3. *The set $\mathcal{E}$ is closed. Hence $(\mathcal{E}, \delta)$ is a complete quasi-metric subspace of $(\mathscr{X}, \delta)$.*

Proof. Let $\{(Y_n, \mu_n) : n \in \mathbb{N}\}$ be a sequence in $\mathcal{E}$ with $(Y_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$. Let us start by proving that $\mathrm{supp}\mu \subseteq Y$. Let us show that $\int \varphi\, d\mu = 0$ for every $\varphi \in \mathcal{C}(X)$ with $\mathrm{supp}\varphi \cap Y = \emptyset$. Take $\epsilon = \rho(\mathrm{supp}\varphi, Y) > 0$ and notice that $\mathrm{supp}\varphi \cap [Y]_\epsilon = \emptyset$. Since, on the other hand, $Y_n \xrightarrow{\delta_H} Y$, for the same value of ϵ, there must exist $N = N(\epsilon)$ such that $Y_n \subseteq [Y]_\epsilon$ whenever $n \geq N$. Hence $\mathrm{supp}\varphi \cap Y_n = \emptyset$ for every $n \geq N$, so that

$$\int \varphi\, d\mu = \lim_{n\to\infty} \int \varphi\, d\mu_n = 0.$$

□

2. Subspaces of $\mathcal{E}$: the doubling property

Let (X, ρ) and d as in Section 1. Even when in most applications of spaces of homogeneous type as models for analytic problems the ρ-balls are open sets, or even when after the above mentioned theorem of Macías and Segovia every quasi-metric is equivalent to another, d^ξ, for which the balls are open and hence Borel sets, it is not difficult to construct a translation invariant quasi-distance $\rho(x, y)$ on $\mathbb{R}$ generating the usual topology and equivalent to $|x - y|$ for which the ρ-balls are not even Lebesgue measurable sets. Hence for a Borel measure μ on (X, ρ) it could happen that the expression $\mu(B_\rho(x, r))$ is not well defined. To avoid this difficulty we shall keep assuming that every ρ-ball is a Borel set.

Let A be a given real number with $A \geq 1$. Let $\mathcal{D}(A)$ be the set of all couples (Y, μ) in $\mathcal{E}$ such that the inequalities

$$0 < \mu(B_\rho(y, 2r)) \leq A\mu(B_\rho(y, r)) \tag{3}$$

hold for every $y \in Y$ and every $r > 0$. Such a couple (Y, μ) is usually called a space of homogeneous type if we understand that the quasi-metric is the one inherited from the environment X.

Theorem 2.1. *Let $\{(Y_n, \mu_n)\}$ be a sequence in $\mathcal{E}$ such that $(Y_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$. If there exists $A \geq 1$ such that $(Y_n, \mu_n) \in \mathcal{D}(A)$ for every n, then there exists A' depending only on A and Λ such that $(Y, \mu) \in \mathcal{D}(A')$.*

Proof. Let φ be the continuous function defined on the non-negative real numbers as $\varphi \equiv 1$ on $[0, 1]$, $\varphi \equiv 0$ on $[2, \infty)$ which is linear in the interval $[1, 2]$. Take $y \in Y$ and $r > 0$. Since $Y_n \xrightarrow{\delta_H} Y$, let us take $y_n \in Y_n$ such that $d(y_n, y) \to 0$ as $n \to \infty$. Then, since $\mathcal{X}_{[0,1]} \leq \varphi$, the following inequality follows easily

$$\mu(B_d(y, 2r)) \leq \int \varphi\left(\frac{d(x, y)}{2r}\right) d\mu(x).$$

Also, for y and r fixed, $\varphi\left(\frac{d(x,y)}{2r}\right)$ is a continuous function of $x \in X$, and since $\mu_n \xrightarrow{w*} \mu$ we have

$$\mu(B_d(y,2r)) \leq \lim_{n\to\infty} \int \varphi\left(\frac{d(x,y)}{2r}\right) d\mu_n(x).$$

Now, since $y_n \to y$ and $\varphi \leq \mathcal{X}_{[0,2]}$ we have the inequalities

$$\begin{aligned}
\mu(B_d(y,2r)) &\leq \liminf_{n\to\infty} \mu_n\left(B_d\left(y,4r\right)\right) \\
&\leq \liminf_{n\to\infty} \mu_n\left(B_d\left(y_n,5r\right)\right) \\
&\leq \liminf_{n\to\infty} A^4\mu_n\left(B_d\left(y_n,\frac{5r}{16}\right)\right) \\
&\leq A^4 \liminf_{n\to\infty} \mu_n\left(B_d\left(y,\frac{r}{2}\right)\right) \\
&\leq A^4 \lim_{n\to\infty} \int \varphi\left(\frac{2d(x,y)}{r}\right) d\mu_n(x) \\
&= A^4 \int \varphi\left(\frac{2d(x,y)}{r}\right) d\mu(x) \\
&\leq A^4\mu(B_d(y,r)).
\end{aligned}$$

Since for every $s > 0$ we have $B_\rho(y,2s) \subseteq B_d(y,(2c_2)^{1/\xi}s^{1/\xi})$ and $B_\rho(y,s) \supseteq B_d(y,c^{1/\xi}s^{1/\xi})$, applying k times the above inequality we obtain

$$\begin{aligned}
\mu(B_\rho(y,2s)) &\leq \mu\left(B_d\left(y,(2c_2)^{1/\xi}s^{1/\xi}\right)\right) \\
&\leq A^{4k}\mu\left(B_d\left(y,c_1^{1/\xi}s^{1/\xi}\right)\right) \\
&\leq A^{4k}\mu(B_\rho(y,s)),
\end{aligned}$$

where $2^k \geq \left(\frac{2c_2}{c_1}\right)^{1/\xi}$ providing a k which only depends on Λ. □

Notice that, since $(\mathcal{E},\delta)$ is complete, given a Cauchy sequence $\{(Y_n,\mu_n)\}$ in $\mathcal{D}(A)$, we have a limit couple $(Y,\mu) \in \mathcal{E}$ for that sequence. The above theorem shows that $(Y,\mu) \in \mathcal{D}(A^4)$ which is a kind of completeness of the doubling classes. Let us also remark that the class $\bigcup_{A\geq 1} \mathcal{D}(A) \subseteq \mathcal{E}$ is not complete. In fact, consider $X = [0,1]$ with ρ the usual distance. Take $Y_n = [0,1]$ for each n and μ_n the measure with density $f_n(t) = n-1+1/n$ on $[0,1/n]$ and $f_n(t) = 1/n$ on $(1/n,1]$. It is easy to see that $\mu_n \xrightarrow{\delta_K} \delta_0$, and that each $(Y_n,\mu_n) \in \mathcal{D}(A_n)$, with $A_n = 2n(n-1+1/n)$ as a possible doubling constant. Actually it is also easy to show that A_n can not be bounded above, since by taking the balls $B(x,r) = B(2/n,1/n)$ we see that $A_n \geq \frac{n^2-n+4}{2}$. Since in each space of homogeneous type atoms are isolated (see [9]), the space $([0,1],|\cdot|,\delta_0)$ can not be a space of homogeneous type.

Let us finally observe that the well know doubling property for the Hausdorff measure of order $\log 2/\log 3$ on the Cantor set is a consequence of the uniform estimates obtained in the introduction and of Theorem 2.1.

3. Density of finite spaces in $\mathcal{D}(A)$

Let us denote by $\mathcal{F}$ the family of all couples (Y,μ) in $\mathcal{E}$ for which Y is finite. In other words

$$\mathcal{F} = \{(Y,\mu) : \operatorname{card}(Y) < \infty\}.$$

Let us observe that each $(Y,\mu) \in \mathcal{F}$ for which $\mu(\{y\}) > 0$ for every $y \in Y$, we have that (Y,μ) belongs to $\mathcal{D}(A)$ for some A. The main result of this paper, which is contained in the next statement, shows that every space $(Y,\mu) \in \mathcal{D}(A)$ can be approximated in the metric δ by a sequence in $\mathcal{F}$ with uniform doubling constant.

Theorem 3.1. *Let (X,ρ) be a compact quasi-metric space for which the ρ-balls are Borel sets. Then, given $A \geq 1$, there exists $A' \geq 1$ depending only on A and on the triangular constant Λ for ρ, such that for every $(Y,\mu) \in \mathcal{D}(A)$ there exists a sequence $(Y_n,\mu_n) \in \mathcal{F} \cap \mathcal{D}(A')$, $n \in \mathbb{N}$, for which*

$$(Y_n,\mu_n) \xrightarrow{\delta} (Y,\mu)$$

as $n \to \infty$.

Before starting with the proof of the theorem we shall state a basic properties of spaces of homogeneous type which is actually contained in the first systematic treatment of spaces of homogeneous type due to R. Coiffman and G. Weiss [5], and reflects the fact that spaces of homogeneous type have finite uniform metric (or Assouad) dimension.

Lemma 3.2. *For each spaces of homogeneous type there exists a geometric constant N such that every r-disperse subset E has at most N^m points in each ball of radius $2^m r$, $m \in \mathbb{N}$.*

Here r-disperse means that the distance between two different points of E is larger than or equal to r.

Proof of Theorem 3.1. It is easy to see that from the above mentioned theorem of Macías and Segovia, we only have to prove the theorem for metric spaces (X,d) checking that, in this particular case, A' only depends on A.

Let d be any one of the metrics provided by the theorem of Macías and Segovia associated to ρ. Since (X,ρ) is compact, we can normalize the distance d in such a way that the d-diameter of X is less that one.

Let $A \geq 1$ be a given constant and take $(Y,d,\mu) \in \mathcal{E}$ such that

$$0 < \mu(B_d(y,2r)) \leq A\mu(B_d(y,r))$$

hold for every $y \in Y$ and every $r > 0$.

We shall combine the idea used by J. M. Wu in [12] and the construction of dyadic sets given by M. Christ in [4] in order to define the approximating sets and the approximating measures.

For each $n \in \mathbb{N} \cup \{0\}$, let $S_n = \{x_{n,k} : 1 \leq k \leq J_n\}$ be a 10^{-n}-net (10^{-n} maximal disperse) in Y with

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$$

Notice that since $\operatorname{diam}(Y) < 1$, the net S_0 contains only one point $x_{0,1}$.

For each $n \in \mathbb{N} \cup \{0\}$, set $P_n = \{T_{n,k} : 1 \leq k \leq J_n\}$ to denote a partition of S_{n+1} satisfying

$$S_{n+1} \cap B_d(x_{n,k}, 10^{-n}/2) \subseteq T_{n,k} \subseteq S_{n+1} \cap B_d(x_{n,k}, 10^{-n}).$$

Let us observe that $\mathrm{card}(S_n) < \infty$ for every $n \in \mathbb{N}$.

Set $\mathcal{A} = \{(n,k) : n \in \mathbb{N} \cup \{0\},\ 1 \leq k \leq J_n\}$, and let us define a partial order $\preceq$ on $\mathcal{A}$ by extending by transitivity the following basic relations

- $(n,k) \preceq (n,k)$ for every $(n,k) \in \mathcal{A}$;
- $(n+1,q) \preceq (n,k)$ if and only if $x_{n+1,q} \in T_{n,k}$.

Let us notice that this order satisfies the following tree properties

(a) $(n_1,k_1) \preceq (n_2,k_2)$ implies $n_2 \leq n_1$;
(b) for every $(n_1,k_1) \in \mathcal{A}$ and every $n_2 \leq j_1$, there exists a unique $1 \leq k_2 \leq J_{n_2}$ such that $(n_1,k_1) \preceq (n_2,k_2)$;
(c) if $(n,k) \preceq (n-1,\ell)$, then $d(x_{n,k}, x_{n-1,\ell}) < 10^{-n+1}$;
(d) if $d(x_{n,k}, x_{n-1,\ell}) < \frac{10^{-n+1}}{2}$, then $(n,k) \preceq (n-1,\ell)$.

Starting from that order and following M. Christ, define

$$Q_k^n = \bigcup_{(i,\ell) \preceq (n,k)} B_d(x_{i,\ell}, 10^{-i-1}).$$

In [4] M. Christ showed the following properties

(a) Q_k^n is an open set for every $(n,k) \in \mathcal{A}$;
(b) $B_d(x_{n,k}, 10^{-n-1}) \subseteq Q_k^n$ for every $(n,k) \in \mathcal{A}$;
(c) $Q_k^n \subseteq B_d(x_{n,k}, 10^{-n+1}/9)$ for every $(n,k) \in \mathcal{A}$;
(d) for each $n \in \mathbb{N}_0$, $Q_k^n \cap Q_\ell^n \neq \emptyset$ implies $k = \ell$;
(e) for every $(n,k) \in \mathcal{A}$ and every $i < n$ there exists a unique $1 \leq \ell \leq J_n$ such that $Q_k^n \subseteq Q_\ell^i$;
(f) if $n \geq i$, for every $1 \leq k \leq J_n$, $1 \leq \ell \leq J_i$ we have that $Q_k^n \subseteq Q_\ell^i$ or $Q_k^n \cap Q_\ell^i = \emptyset$;
(g) $\mu\left(Y \setminus \bigcup_{1 \leq k \leq J_n} Q_k^n\right) = 0$, for every $n \in \mathbb{N}_0$;
(h) $\mu(Q_k^n) = \sum_{i:(\ell,i) \preceq (n,k)} \mu(Q_i^\ell)$, for every $n \in \mathbb{N} \cup \{0\}$, $\ell \geq n+1$, and $1 \leq k \leq J_n$.

The next step is the construction of measures μ_n on Y with total mass on S_n. Let us define

$$\mu_n(\{x_{n,k}\}) = \mu(Q_k^n),$$

for every $n \in \mathbb{N} \cup \{0\}$ and $1 \leq k \leq J_n$. Notice that $\mu_n(S_n) = 1$ for every n. Let us check that $\mu_n \xrightarrow{w*} \mu$. Take a continuous function φ on Y, and let $\epsilon > 0$ be given. Since Y is compact, φ es uniformly continuous, hence there exists $\eta > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$, for every $x, y \in Y$ tales que $d(x,y) < \eta$. Let us observe that

$$\int_Y \varphi\, d\mu_n = \sum_{k=1}^{J_n} \varphi(x_{n,k}) \mu(Q_k^n),$$

and on the other hand

$$\int_Y \varphi\, d\mu = \sum_{k=1}^{J_n} \int_{Q_k^n} \varphi\, d\mu.$$

Thus

$$\left|\int_Y \varphi\, d\mu_n - \int_Y \varphi\, d\mu\right| \leq \sum_{k=1}^{J_n} \int_{Q_k^n} |\varphi(x_{n,k}) - \varphi(x)|\, d\mu(x) < \epsilon,$$

choosing n large enough to get $10^{-n+1}/9 < \eta$.

Since $\delta_H(S_n, Y) \leq 10^{-n}$, we have that $S_n \xrightarrow{\delta_H} Y$. So that $(S_n, \mu_n) \xrightarrow{\delta} (Y, \mu)$.

It only remains to prove that (S_n, d, μ_n) is a uniform family os spaces of homogeneous type. The proof will be based on the following three properties.

First: If we define for $\ell \geq n+1$

$$T_{n,k}^{\ell} = \{x_{\ell,i} : (\ell, i) \preceq (n,k)\},$$

from (h) we have,

$$\mu_\ell(T_{n,k}^\ell) = \sum_{i: x_{\ell,i} \in T_{n,k}^\ell} \mu_\ell(\{x_{\ell,i}\}) = \sum_{i:(\ell,i)\preceq(n,k)} \mu(Q_i^\ell) = \mu_n(\{x_{n,k}\}). \tag{4}$$

for every $n \in \mathbb{N} \cup \{0\}$, $1 \leq k \leq J_n$ and $\ell \geq n+1$.

Second: For $x_{n,k} \in S_n$ and $x_{n+1,\ell}$ with $(n+1, \ell) \preceq (n,k)$ we have $d(x_{n,k}, x_{n+1,\ell}) < 10^{-n}$. Then $Q_k^n \subseteq B_d(x_{n+1,\ell}, 19\,10^{-n}/9)$. So that

$$\begin{aligned} \mu_n(\{x_{n,k}\}) &= \mu(Q_k^n) \\ &\leq \mu(B_d(x_{n+1,\ell}, 19\,10^{-n}/9)) \\ &\leq A^8 \mu(B_d(x_{n+1,\ell}, 10^{-n-2})) \\ &\leq A^8 \mu(Q_\ell^{n+1}) \\ &= A^8 \mu_{n+1}(\{x_{n+1,\ell}\}). \end{aligned} \tag{5}$$

Third: If $x_{n,k}$ and $x_{n,i}$ are points in S_n such that $d(x_{n,k}, x_{n,i}) < 2\,10^{-k+2}$, then

$$\mu_n(\{x_{n,k})\} \leq A^{11} \mu_n(\{x_{n,i}\}). \tag{6}$$

Let us prove that (S_n, d, μ_n) is a uniform family of spaces of homogeneous type. For $n = 0$ we have $\mu_0\left(B_d(x_{0,1}, 2r)\right) = \mu_0\left(B_d(x_{0,1}, r)\right) = \mu_0(\{x_{0,1}\}) = 1$ for every $r > 0$, and the result is trivial. Let us take $n \geq 1$ fixed, $x = x_{n,j} \in S_n$ and $r > 0$. We shall divide our analysis according to the relation between r and n:

i. $0 < r \leq 10^{-n}/2$;
ii. $10^{-n}/2 < r \leq 3\,10^{-n+1}$;
iii. $r > 3\,10^{-n+1}$.

Case i: $0 < r \leq 10^{-n}/2$. Since S_n is 10^{-n}-disperse, we have $B_d(x, 2r) \cap S_n = B_d(x, r) \cap S_n = \{x\}$, and again the result is trivial.

Case ii: $10^{-n}/2 < r \leq 3\,10^{-n+1}$. Let $\mathcal{I}$ be the set defined as

$$\mathcal{I} = \{q : x_{n-1,q} \in B_d(x, 23r)\}.$$

Applying Lemma 3.2 we have

$$\begin{aligned}
\operatorname{card}(\mathcal{I}) &= \operatorname{card}\left(S_{n-1} \cap B_d(x, 23r)\right) \\
&\leq \operatorname{card}\left(S_{n-1} \cap B_d(x, 69\,10^{-n+1})\right) \\
&\leq \operatorname{card}\left(S_{n-1} \cap B_d(x, 2^7 10^{-n+1})\right) \\
&\leq N^7. \qquad (7)
\end{aligned}$$

Claim 3.3.

$$S_n \cap B_d(x, 3r) \subseteq \bigcup_{q \in \mathcal{I}} T_{n-1,q}.$$

Poof of Claim 3.3. Take $x_{n,i} \in B_d(x, 3r)$, and let q be the unique index such that $x_{n,i} \in T_{n-1,q}$. Let us prove that $q \in \mathcal{I}$. In fact, since

$$T_{n-1,q} \subseteq S_n \cap B_d(x_{n-1,q}, 10^{-n+1}),$$

we have

$$\begin{aligned}
d(x_{n-1,q}, x) &\leq d(x_{n-1,q}, x_{n,i}) + d(x_{n,i}, x) \\
&< 10^{-n+1} + 3r \\
&< 20r + 3r \\
&< 23r,
\end{aligned}$$

which proves the claim.

Let p be such that $x \in T_{n-1,p}$. If $q \in \mathcal{I}$, then

$$\begin{aligned}
d(x_{n-1,q}, x_{n-1,p}) &\leq d(x_{n-1,q}, x) + d(x, x_{n-1,p}) \\
&< 23r + 10^{-n+1} \\
&< 69\,10^{-n+1} + 10^{-n+1} \\
&< 2\,10^{-n+3},
\end{aligned}$$

and we can apply (6) to get

$$\mu_{n-1}\left(\{x_{n-1,q}\}\right) \leq A^{11} \mu_{n-1}\left(\{x_{n-1,p}\}\right), \quad \text{for every } q \in \mathcal{I}.$$

Then, from (4), (7) and (5) we have

$$\begin{aligned}
\mu_n(B_d(x, 3r)) &\leq \sum_{q \in \mathcal{I}} \mu_n\left(T_{n-1,q}\right) \\
&= \sum_{q \in \mathcal{I}} \mu_{n-1}(\{x_{n-1,q}\}) \\
&\leq \sum_{q \in \mathcal{I}} A^{11} \mu_{n-1}(\{x_{n-1,p}\}) \\
&\leq N^7 A^{11} \mu_{n-1}(\{x_{n-1,p}\}) \\
&\leq N^7 A^{19} \mu_n(\{x\}) \\
&\leq N^7 A^{19} \mu_n(B_d(x, r)).
\end{aligned}$$

Hence for every $10^{-n}/2 < r \leq 3\,10^{-n+1}$ we have

$$\mu_n(B_d(x, 2r)) \leq \mu_n(B_d(x, 3r)) \leq N^7 A^{19} \mu_n(B_d(x, r)).$$

Case iii: $r > 3\,10^{-n+1}$. Since we can assume $r \leq 1$, this case is only possible if $n \geq 2$. Let $0 < \ell \leq n-1$ such that $10^{-\ell} < r/3 \leq 10^{-\ell+1}$, and define the set $\mathcal{J} = \{j : x_{\ell,j} \in S_\ell \cap B_d(x, 3r)\}$. Then

$$S_n \cap B_d(x, 2r) \subseteq \bigcup_{j\in\mathcal{J}} T^n_{\ell,j}\,.$$

In fact, take $x_{n,i} \in B_d(x, 2r)$ and $x_{\ell,j}$ such that $x_{n,i} \in T^n_{\ell,j}$. Then

$$\begin{aligned} d(x_{\ell,j}, x) &\leq d(x_{\ell,j}, x_{n,i}) + d(x_{n,i}, x) \\ &< \frac{10^{-\ell+1}}{9} + 2r \\ &< \frac{10}{27} r + 2r \\ &< 3r, \end{aligned}$$

and thus $j \in \mathcal{J}$. Hence we have

$$\mu_n(B_d(x, 2r)) \leq \sum_{j\in\mathcal{J}} \mu_n\left(T^n_{\ell,j}\right) = \sum_{j\in\mathcal{J}} \mu_\ell(\{x_{\ell,j}\}) = \mu_\ell(B_d(x, 3r)). \tag{8}$$

On the other hand

$$\bigcup_{x_{\ell,i}\in B_d(x,r/2)} T^n_{\ell,i} \subseteq S_n \cap B_d(x, r).$$

In fact, if $x_{\ell,i} \in B_d(x, r/2)$ and $x_{n,p} \in T^n_{\ell,i}$, then

$$\begin{aligned} d(x, x_{n,p}) &\leq d(x, x_{\ell,i}) + d(x_{\ell,i}, x_{n,p}) \\ &< \frac{r}{2} + \frac{10^{-\ell+1}}{9} \\ &< \frac{r}{2} + \frac{10}{27} r \\ &< r. \end{aligned}$$

By the above inclusion we obtain

$$\mu_n(B_d(x, r)) \geq \sum_{x_{\ell,i}\in B_{r/2}} \mu_n\left(T^n_{\ell,i}\right) = \sum_{x_{\ell,i}\in B_{r/2}} \mu_\ell(\{x_{\ell,i}\}) = \mu_\ell(B_{r/2}), \tag{9}$$

where $B_{r/2} = B_d(x, r/2)$.

Claim 3.4. $\mu_\ell(B_d(x, 3r)) \leq A^{19} N^8 \mu_\ell(B_d(x, r/2))$.

Proof of Claim 3.4. If we define

$$\mathcal{I} = \{q : x_{\ell,q} \in B_d(x, 19r/3)\},$$
$$\mathcal{J} = \{j : x_{\ell-1,j} \in B_d(x, 19r/3)\},$$

we have

$$\begin{aligned} \operatorname{card}(\mathcal{I}) &\leq \operatorname{card}\left(S_\ell \cap B_d(x, 19\;10^{-\ell+1})\right) \\ &\leq \operatorname{card}\left(S_\ell \cap B_d(x, 256\;10^{-\ell})\right) \\ &\leq N^8. \end{aligned}$$

Notice that

$$S_\ell \cap B_d(x,3r) \subseteq \bigcup_{j\in\mathcal{J}} T_{\ell-1,j}.$$

In order to prove the above inclusion, take $x_{\ell,i} \in B_d(x,3r)$ and $x_{\ell-1,j}$ such that $x_{\ell,i} \in T_{\ell-1,j}$. Then

$$\begin{aligned} d(x_{\ell-1,j},x) &\leq d(x_{\ell-1,j},x_{\ell,i}) + d(x_{\ell,i},x) \\ &< 10^{-\ell+1} + 3r \\ &< 10r/3 + 3r \\ &= 19r/3. \end{aligned}$$

Hence $j \in \mathcal{J}$.

On the other hand, if $x \in T^n_{\ell,p}$ and $q \in \mathcal{I}$ we have

$$\begin{aligned} d(x_{\ell,p},x_{\ell,q}) &\leq d(x_{\ell,p},x) + d(x,x_{\ell,q}) \\ &< \frac{10^{-\ell+1}}{9} + \frac{19}{3}\,r \\ &< \frac{10^{-\ell+1}}{9} + 19\;10^{-\ell+1} \\ &= \frac{172}{9}\;10^{-\ell+1} \\ &< 2\;10^{-\ell+2}, \end{aligned}$$

and by applying (6) we obtain

$$\mu_\ell(\{x_{\ell,q}\}) \leq A^{11}\mu_\ell(\{x_{\ell,p}\}), \quad \text{for every } q \in \mathcal{I}.$$

Finally, if we define

$$\mathcal{A} = \{i : \text{ there exists } j \in \mathcal{J} \text{ satisfying } x_{\ell,i} = x_{\ell-1,j}\}.$$

from the considerations above we have

$$\begin{aligned} \mu_\ell(B_d(x,3r)) &\leq \sum_{j\in\mathcal{J}} \mu_\ell\left(T_{\ell-1,j}\right) \\ &= \sum_{j\in\mathcal{J}} \mu_{\ell-1}\left(\{x_{\ell-1,j}\}\right) \\ &\leq A^8 \sum_{i\in\mathcal{A}} \mu_\ell(\{x_{\ell,i}\}) \\ &\leq A^8 \sum_{i\in\mathcal{I}} \mu_\ell(\{x_{\ell,i}\}) \\ &\leq A^{19} \sum_{i\in\mathcal{I}} \mu_\ell(\{x_{\ell,p}\}) \\ &\leq A^{19}N^8\mu_\ell(\{x_{\ell,p}\}) \\ &\leq A^{19}N^8\mu_\ell\left(B_d(x,r/2)\right), \end{aligned}$$

where the last inequality holds since

$$d(x_{\ell,p},x) < \frac{10^{-\ell+1}}{9} < \frac{10}{27}\,r < \frac{r}{2}$$

because $x \in T^n_{\ell,p}$. This proves the claim.

Then, from (8), Claim 3.4 and (9), we can conclude that

$$\begin{aligned}\mu_n(B_d(x,2r)) &\leq \mu_\ell(B_d(x,3r))\\ &\leq A^{19}N^8\,\mu_\ell(B_d(x,r/2))\\ &\leq A^{19}N^8\,\mu_n(B_d(x,r)),\end{aligned}$$

for every $x \in S_n$ and every $r > 3\,10^{-n+1}$.

So that we get the desired inequality for every $x \in S_n$, every $r > 0$ and every $n \in \mathbb{N} \cup \{0\}$ with $A' = A^{19}N^8$. □

References

1. Cecilia Åkerlund-Biström, *A generalization of the Hutchinson distance and applications*, Random Comput. Dynam. **5** (1997), no. 2-3, 159–176. MR **MR1460463 (99e:**28011)
2. L. Ambrosio, M. Miranda, Jr., and D. Pallara, *Special functions of bounded variation in doubling metric measure spaces*, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, Quad. Mat., vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 1–45. MR **MR2118414 (2005j:**49036)
3. Patrick Billingsley, *Convergence of probability measures*, Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999. MR **MR1700749 (2000e:**60008)
4. Michael Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), no. 2, 601–628. MR **MR1096400 (92k:**42020)
5. Ronald R. Coifman and Guido Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Springer-Verlag, Berlin, 1971, Étude de certaines intégrales singulières, Lecture Notes in Mathematics, Vol. 242. MR MR0499948 (58 #17690)
6. Kenneth Falconer, *Techniques in fractal geometry*, John Wiley & Sons Ltd., Chichester, 1997. MR **99f:**28013
7. Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR MR0257325 (41 #1976)
8. A. S. Kravchenko, *Completeness of the space of separable measures in the Kantorovich-Rubinshteĭn metric*, Sibirsk. Mat. Zh. **47** (2006), no. 1, 85–96. MR **MR2215298 (2007b:**28003)
9. Roberto A. Macías and Carlos Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), no. 3, 257–270. MR **MR546295 (81c:**32017a)
10. Umberto Mosco, *Variational fractals*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 3-4, 683–712 (1998), Dedicated to Ennio De Giorgi. MR **MR1655537 (99m:**28023)
11. H. L. Royden, *Real analysis*, The Macmillan Co., New York, 1963. MR MR0151555 (27 #1540)
12. Jang-Mei Wu, *Hausdorff dimension and doubling measures on metric spaces*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1453–1459. MR **99h:**28016
13. Po-Lam Yung, *Doubling properties of self-similar measures*, Indiana Univ. Math. J. **56** (2007), no. 2, 965–990. MR MR2317553