

Absorbing Boundary Conditions for Compressible Flow at Low Mach Number and Free Surface Flows with Level-Set

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1 Governing Equations

$$\begin{aligned} \rho \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \\ \frac{1}{c^2} \frac{\partial p}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \end{aligned} \quad (1)$$

This can be cast as an advective diffusive system of equations. In order to simplify the absorbing boundary conditions we will make the following assumptions at the region close to the outlet boundary

- The convective term ($\rho \mathbf{u} \cdot \nabla \mathbf{u}$) is neglected (low Mach assumption).
- Changes in density are small.

In order to derive absorbing boundary conditions we don't take into account the diffusive and source term, then the system of equations is

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \cdot \nabla \mathbf{U} = 0 \quad (2)$$

where $\mathbf{U} = [u, v, w, \tilde{p}]$ is the state vector, $\tilde{p} = p/\rho$, and $\mathbf{A} = [\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z]$, where each of the \mathbf{A}_j are matrices of 4×4 , and

$$\mathbf{A} \cdot \mathbf{v} = v_x \mathbf{A}_x + v_y \mathbf{A}_y + v_z \mathbf{A}_z, \quad (3)$$

for any vector \mathbf{v} . The Jacobians are defined in compact form as

$$\mathbf{A} \cdot \mathbf{k} = \begin{bmatrix} \mathbf{0} & \mathbf{k} \\ c^2 \mathbf{k}^T & 0 \end{bmatrix}, \quad (4)$$

for an arbitrary vector \mathbf{k} . For instance \mathbf{A}_x is obtained by replacing $\mathbf{k} = [1, 0, 0]^T$ in (4) and we get

$$A_x = \mathbf{A} \cdot [1, 0, 0]^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c^2 & 0 & 0 & 0 \end{bmatrix}, \quad (5)$$

2 Absorbing boundary conditions

Given a boundary with unit length normal vector $\hat{\mathbf{n}}$, we can build the absorbing boundary conditions in terms of the eigen-decomposition of $\mathbf{A} \cdot \hat{\mathbf{n}}$, given by

$$\mathbf{A} \cdot \hat{\mathbf{n}} = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{n}} \\ c^2 \hat{\mathbf{n}}^T & 0 \end{bmatrix}. \quad (6)$$

The four eigenvalues and eigenvectors λ_j, \mathbf{v}_j of this matrix are

$$\mathbf{v}_{1,2} = \begin{bmatrix} \hat{\mathbf{t}}_{1,2} \\ 0 \end{bmatrix}, \quad \lambda_{1,2} = 0 \quad (7)$$

$$\mathbf{v}_{3,4} = \begin{bmatrix} \hat{\mathbf{n}} \\ \pm c \end{bmatrix}, \quad \lambda_{3,4} = \pm c \quad (8)$$

$$(9)$$

where the $\hat{\mathbf{t}}_{1,2}$ are two unit length vectors tangent to the surface and mutually orthogonal ($\hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 = 0$), and then orthogonal to $\hat{\mathbf{n}}$. So that, in a given boundary we have 1 incoming (i.e. negative) characteristic, λ_4 , and two neutral ($\lambda_{1,2}$).

The matrix of eigenvectors is

$$\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4] = \begin{bmatrix} \hat{\mathbf{t}}_1 & \hat{\mathbf{t}}_2 & \hat{\mathbf{n}} & \hat{\mathbf{n}} \\ 0 & 0 & c & -c \end{bmatrix} \quad (10)$$

and its inverse is

$$\mathbf{W} = \mathbf{V}^{-1} = \begin{bmatrix} \hat{\mathbf{t}}_1 & 0 \\ \hat{\mathbf{t}}_2 & 0 \\ 1/2\hat{\mathbf{n}} & 1/(2c) \\ 1/2\hat{\mathbf{n}} & -1/(2c) \end{bmatrix}, = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}. \quad (11)$$

The absorbing boundary conditions corresponds then to imposing

$$w_j \cdot (\mathbf{U} - \mathbf{U}_{\text{ref}}) = 0, \quad (12)$$

for the w_j corresponding to the incoming (negative) eigenvalues. We also do the same for the neutral (null) eigenvalues.

$$\begin{aligned} \hat{\mathbf{t}}_{1,2} \cdot \mathbf{u} &= 0, \\ \rho c \hat{\mathbf{n}} \cdot \mathbf{u} - (p - p_{\text{ref}}) &= 0. \end{aligned} \quad (13)$$

This last can be put in matricial form as follows,

$$\begin{bmatrix} \hat{\mathbf{t}}_1^T \\ \hat{\mathbf{t}}_2^T \\ \hat{\mathbf{n}}^T \end{bmatrix} \mathbf{u} - \frac{(p - p_{\text{ref}})}{\rho c} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}. \quad (14)$$

The first matrix is orthogonal, so that multiplying by its transpose we get

$$[\hat{\mathbf{t}}_1 \quad \hat{\mathbf{t}}_2 \quad \hat{\mathbf{n}}] \left\{ \begin{bmatrix} \hat{\mathbf{t}}_1 \\ \hat{\mathbf{t}}_2 \\ \hat{\mathbf{n}} \end{bmatrix} \mathbf{u} - \frac{(p - p_{\text{ref}})}{\rho c} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathbf{0}, \quad (15)$$

and finally,

$$\mathbf{u} - \frac{(p - p_{\text{ref}})}{\rho c} \hat{\mathbf{n}} = \mathbf{0}. \quad (16)$$

3 Using penalization

If it is not possible to impose directly this equations, then it can be do by Lagrange multipliers or penalization. In the last case you should add

$$K[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4] \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_4 \end{bmatrix} (\mathbf{U} - \mathbf{U}_{\text{ref}}) = K\mathbf{H}(\mathbf{U} - \mathbf{U}_{\text{ref}}) = 0. \quad (17)$$

where K is a large penalization parameter. Note that put in this way the coefficients added to the diagonal contributions are positive.

An explicit expression for \mathbf{H} can be found,

$$\mathbf{H} = \begin{bmatrix} \hat{\mathbf{t}}_1 & \hat{\mathbf{t}}_2 & \hat{\mathbf{n}} \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} \hat{\mathbf{t}}_1^T & 0 \\ \hat{\mathbf{t}}_2^T & 0 \\ \hat{\mathbf{n}}^T & -1/c \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{3 \times 3} & -\hat{\mathbf{n}}/c \\ -c\hat{\mathbf{n}}^T & 1 \end{bmatrix} \quad (18)$$

Take note that this is posed in terms of \mathbf{U} , where the last component is p/ρ . This has to be taken into account when adding these terms to the FEM equations.

Moreover, if the condition number is too high or you want to get better absorption then it is possible to use a distributed absorbing term by adding a distributed term of the form $\kappa(\mathbf{x})\mathbf{H}(\mathbf{U} - \mathbf{U}_{\text{ref}}) = 0$ to the system of equations. Note that this term contributes to the momentum equations as well as to the continuity equation. $\kappa(x)$ is a parameter controlling locally the degree of absorption. it may be put constant in a certain slab near the boundary or growing linearly in the gap, from 0 at the inner boundary of the gap to a given high value in the outer boundary.

$$\begin{aligned} \rho \frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p + \kappa(\rho \mathbf{u} - \frac{1}{c}(p - p_{\text{ref}})\hat{\mathbf{n}}) &= \mathbf{f} \\ \frac{1}{c^2} \frac{\partial p}{\partial t} + \nabla \cdot \rho \mathbf{u} + \frac{\kappa(x)}{c^2} [(p - p_{\text{ref}}) - \rho c \mathbf{u} \cdot \hat{\mathbf{n}}] &= 0 \end{aligned} \quad (19)$$

3.1 Notes

- These boundary conditions must be applied on the whole boundary.
- It is better if the boundary is smooth (for instance a sphere). If not, the absorption is lower but it is expected that the condition will improve the computation.

4 Algorithm for computing the tangents

Given the normal vector $\hat{\mathbf{n}}$, we can take any vector \mathbf{w} and compute

$$\begin{aligned} \hat{\mathbf{t}}_1 &= \frac{\mathbf{w} \times \hat{\mathbf{n}}}{\|\mathbf{w} \times \hat{\mathbf{n}}\|}, \\ \hat{\mathbf{t}}_2 &= \frac{\hat{\mathbf{t}}_1 \times \hat{\mathbf{n}}}{\|\hat{\mathbf{t}}_1 \times \hat{\mathbf{n}}\|}. \end{aligned} \quad (20)$$

In order to avoid singularities we should avoid that \mathbf{w} be not parallel to $\hat{\mathbf{n}}$. In order to ensure that we take $\mathbf{w} = \hat{\mathbf{e}}_j$, i.e. the the unit vector along axis j , where j is the smaller component of $\hat{\mathbf{n}}$ in absolute value.

Note, however, that in the form posed above, there is no need to compute explicitly the tangents.

5 Absorbing layer for free surface flows

We develop here an Absorbing Layer (AL) for the simulation of free surface (FS) flows with methods like Level-Set (LS), Volume-Of-Fluid (VOF), or Particle Finite Element Method (PFEM) combined with the resolution of the Navier-Stokes equations.

The idea is that first, we find the AL for the shallow water (SW) approximation, which is straightforward. Then we discuss how to apply this to the FS flows.

5.1 AL for the shallow water equations

Section §5.1 assumes that the state vector is represented as $\mathbf{U} = [h, \mathbf{u}]^T$, while section §5.5 assumes the converse, i.e. $\mathbf{U} = [\mathbf{u}, h]^T$ (which is the ordering used in PETSc-FEM).

We assume that in the simulation of the FS flow, near the boundary the flow can be approximated by the shallow water equations, which are

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{G}, \quad (21)$$

where

$$\begin{aligned} \mathbf{U} &= [h, u]^T, \\ \mathbf{F}(\mathbf{U}) &= [hu, \frac{1}{2}u^2 + gh]^T, \\ \mathbf{G} &= [0, -g\frac{\partial H}{\partial x}]^T. \end{aligned} \quad (22)$$

where

- $h(x)$ is the water height,
- $u(x)$ is the mean horizontal velocity (averaged in the vertical z direction),
- $\mathbf{F}(\mathbf{U})$ are the advective fluxes,
- g is gravity,
- $H(x)$ is the height of the bottom with respect to a fixed reference plane.
- \mathbf{G} is a source term to the momentum equation due to the changing slope of the bottom surface. As we will assume that the near the outlet boundary the surface bottom is plane, then we can discard this term.

The Jacobian, of the flux is then

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} u & h \\ g & u \end{bmatrix}. \quad (23)$$

The eigenvalue decomposition is

$$\begin{aligned}
\mathbf{A} &= \mathbf{V}^{-1} \mathbf{\Lambda} \mathbf{V}, \\
\mathbf{\Lambda} &= \text{diag} [u + c, u - c], \\
c &= \sqrt{gh}, \\
\mathbf{V} &= \begin{bmatrix} h & h \\ c & -c \end{bmatrix}, \\
\mathbf{W} &= \mathbf{V}^{-1} = 1/2 \begin{bmatrix} 1/h & 1/c \\ 1/h & -1/c \end{bmatrix}.
\end{aligned} \tag{24}$$

We assume that the flow is subcritical at the outlet boundary, then we have one incoming characteristic ($u - c$), and then the penalization term (see (17)) is

$$\begin{aligned}
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + K \mathbf{H} (\mathbf{U} - \mathbf{U}_{\text{ref}}) &= \mathbf{G}, \\
\mathbf{U}_{\text{ref}} &= [h_0, u_0],
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\mathbf{H} &= \mathbf{V} \text{diag} [0, u_0 - c_0] \mathbf{W}, \\
&= \mathbf{v}_2 (u_0 - c_0) \mathbf{w}_2^T, \\
&= \begin{bmatrix} h_0 \\ -c_0 \end{bmatrix} (u_0 - c_0) \begin{bmatrix} 1/h_0 & -1/c_0 \end{bmatrix}, \\
&= (u_0 - c_0) \begin{bmatrix} 1 & -h_0/c_0 \\ -c_0/h_0 & 1 \end{bmatrix}.
\end{aligned} \tag{26}$$

where $c_0 = \sqrt{gh_0}$. The modified shallow water equations are then

$$\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (uh) + K h_0 (u_0 - c_0) \left\{ \frac{(h - h_0)}{h_0} - \frac{(u - u_0)}{c_0} \right\} &= 0, \\
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (1/2 u^2 + gh) - K c_0 (u_0 - c_0) \left\{ \frac{(h - h_0)}{h_0} - \frac{(u - u_0)}{c_0} \right\} &= 0
\end{aligned} \tag{27}$$

5.2 Implementation of the shallow water AL in the NS-FS model

Now, in order to apply this AL to the NS-FS equations as they are commonly solved with the LS, VOF we must identify how to identify the u and h quantities from the NS model and then how to apply the terms in a distributed form to the NS equations.

Remember that the u in the SW equations (from here on u_{SW}) is the vertical average of the x -component u of velocity that, which would be obtained with the NS model

$$u_{\text{SW}} = \frac{1}{h} \int_{z=0}^h u \, dz. \tag{28}$$

As an approximation we will assume that the perturbation of the waves near the outlet boundary is small, so that $u_{\text{SW}} \approx u_{\text{SW}} = u$. With respect to u_0 we can take an average (assuming the mean conditions are steady) or either we can take as u_0 the velocity value at the previous time step.

The value of h is harder to obtain from the NS model. First we will recall the shape of a wave, from potential theory

$$\begin{aligned}\phi(x, z, t) &= A \exp\{i(kx - \omega t)\} \frac{\cosh(kz)}{\cosh(kh_0)}, \\ u(x, z, t) &= \frac{\partial \phi}{\partial x} = ikA \exp\{i(kx - \omega t)\} \frac{\cosh(kz)}{\cosh(kh_0)}, \\ w(x, z, t) &= \frac{\partial \phi}{\partial z} = kA \exp\{i(kx - \omega t)\} \frac{\cosh(kz)}{\cosh(kh_0)},\end{aligned}\quad (29)$$

where ϕ is the velocity potential, k is the wave number, A the wave amplitude, and ω the angular frequency of the wave. It is assumed that $z = 0$ at the bottom surface, and h_0 is the mean height of water. These expressions are complex, we assume that always we have to take the real part of them.

The pressure can be obtained from the unsteady Bernoulli equation

$$\frac{p}{\rho} + gz + \frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 = \text{cnst} = \frac{p_{\text{atm}}}{\rho} + gz_0 + \frac{1}{2}u_{\infty}^2. \quad (30)$$

We assume small amplitude waves, so that we can neglect the quadratic velocity terms, and then we arrive to

$$\frac{p - p_{\text{atm}}}{\rho} + gz = i\omega A \exp\{i(kx - \omega t)\} \frac{\cosh(kz)}{\cosh(kL)}. \quad (31)$$

The position of the free surface is found by requiring that $p = p_{\text{atm}}$ so that we obtain

$$\eta = h - h_0 = \frac{i\omega A}{g} \exp\{i(kx - \omega t)\}. \quad (32)$$

We see then, that the expressions for the elevation perturbation η and p are very similar. For shallow water waves ($kh \ll 1$) the depth factor is unity and then they are proportional, i.e.

$$\frac{p - p_{\text{atm}}}{\rho g} + (z - z_0) \approx \eta = h - h_0 \quad (33)$$

so that we can take the $(p - p_{\text{atm}})/\rho g + (z - h_0)$ as a replacement for $h - h_0$ in the absorbing term.

5.3 The optimal absorbing matrix coefficients

We have a system of 1D advective linear systems of equations with constant coefficients

$$\mathbf{C} \frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{H} \mathbf{U} = 0. \quad (34)$$

We diagonalize \mathbf{A} as follows

$$\mathbf{A} \mathbf{S} = \mathbf{C} \mathbf{S} \mathbf{\Lambda}, \quad (35)$$

where \mathbf{S} is the matrix of eigenvectors, and $\mathbf{\Lambda}$ a diagonal matrix whose diagonal entries are the corresponding eigenvalues. Since we assumed that the system is advective such a decomposition must exist and the eigenvalues must be real. We further assume that the eigenvalues are sorted, such that the first n^- are negative (left-going), and the rest n^+ are positive (right-going), with $n^+ + n^- = n$ the dimension of the state vector \mathbf{U} . Changing to variables

$$\mathbf{V} = \mathbf{S}^{-1} \mathbf{U}, \quad (36)$$

we arrive at the decoupled equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{\Lambda} \frac{\partial \mathbf{V}}{\partial x} + \tilde{\mathbf{H}} \mathbf{V} = 0. \quad (37)$$

with $\tilde{\mathbf{H}} = \mathbf{S}^{-1} \mathbf{C}^{-1} \mathbf{H} \mathbf{S}$. We decompose \mathbf{V} in the right and left-going components

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}^- \\ \mathbf{V}^+ \end{bmatrix}. \quad (38)$$

and (37) transforms to

$$\begin{aligned} \frac{\partial \mathbf{V}^-}{\partial t} + \mathbf{\Lambda}^- \frac{\partial \mathbf{V}^-}{\partial x} + \tilde{\mathbf{H}}_{11} \mathbf{V}^- + \tilde{\mathbf{H}}_{12} \mathbf{V}^+ &= 0, \\ \frac{\partial \mathbf{V}^+}{\partial t} + \mathbf{\Lambda}^+ \frac{\partial \mathbf{V}^+}{\partial x} + \tilde{\mathbf{H}}_{21} \mathbf{V}^- + \tilde{\mathbf{H}}_{22} \mathbf{V}^+ &= 0. \end{aligned} \quad (39)$$

We assume that this equations are valid in the AL $0 \leq x \leq L$.

Regarding the boundary conditions, a periodic wave $\mathbf{V}_0^+ = \mathbf{V}_{0,\text{in}}^+ e^{i\omega t}$ is injected through the right-going components at $x = 0$. This is propagated to $x = L$, partially reflected and comes back through the left-going component \mathbf{V}^- . As a measure of the transparency of the AL we *sense* the resulting left-going components at $x = 0$, i.e. $\mathbf{V}^-(0)$. The boundary condition at $x = L$ is of the form

$$\mathbf{V}^- = \mathbf{D} \mathbf{V}^+, \quad \text{at } x = L. \quad (40)$$

The periodic solution to (34) can be found by operational methods. First we propose a periodic solution in time as

$$\mathbf{U}(x, t) = \hat{\mathbf{U}}(x) e^{i\omega t}. \quad (41)$$

Replacing in (34) we obtain an equation that does not depend on time

$$i\omega \hat{\mathbf{U}} + \mathbf{A} \frac{\partial \hat{\mathbf{U}}}{\partial x} + \mathbf{H} \hat{\mathbf{U}} = 0. \quad (42)$$

whose solution is of the form

$$\hat{\mathbf{U}}(x) = \exp\{-\mathbf{A}^{-1}(i\omega\mathbf{C} + \mathbf{H})x\} \hat{\mathbf{U}}(0). \quad (43)$$

so that

$$\hat{\mathbf{U}}(0) = \mathbf{B}\hat{\mathbf{U}}(L), \quad (44)$$

with

$$\mathbf{B} = \exp\{\mathbf{A}^{-1}(i\omega\mathbf{C} + \mathbf{H})L\}. \quad (45)$$

where the $\exp(\cdot)$ must be interpreted in the sense of function of a matrix, based on an eigenvalue decomposition. Transforming to $\hat{\mathbf{V}} = \mathbf{S}^{-1}\hat{\mathbf{U}}$ variables

$$\hat{\mathbf{V}}(0) = \tilde{\mathbf{B}}\hat{\mathbf{V}}(L). \quad (46)$$

with

$$\tilde{\mathbf{B}} = \mathbf{S}^{-1}\mathbf{B}\mathbf{S}. \quad (47)$$

We have the boundary conditions

$$\begin{aligned} \hat{\mathbf{V}}_0^+ &= \mathbf{V}_{0,\text{in}}^+, \\ \hat{\mathbf{V}}_L^- &= \mathbf{D}\hat{\mathbf{V}}_L^+. \end{aligned} \quad (48)$$

Now decomposing (46) in components we have

$$\begin{aligned} \hat{\mathbf{V}}_0^- &= \tilde{\mathbf{B}}_{11}\hat{\mathbf{V}}_L^- + \tilde{\mathbf{B}}_{12}\hat{\mathbf{V}}_L^+ = (\tilde{\mathbf{B}}_{11}\mathbf{D} + \tilde{\mathbf{B}}_{12})\hat{\mathbf{V}}_L^+, \\ \hat{\mathbf{V}}_0^+ &= \tilde{\mathbf{B}}_{21}\hat{\mathbf{V}}_L^- + \tilde{\mathbf{B}}_{22}\hat{\mathbf{V}}_L^+ = (\tilde{\mathbf{B}}_{21}\mathbf{D} + \tilde{\mathbf{B}}_{22})\hat{\mathbf{V}}_L^+. \end{aligned} \quad (49)$$

Now, with (49) and (48) we can obtain an explicit expression for the reflected wave amplitude $\hat{\mathbf{V}}_0^-$ in terms of the incoming wave amplitude $\hat{\mathbf{V}}_0^+$.

$$\begin{aligned} \hat{\mathbf{V}}_0^- &= (\tilde{\mathbf{B}}_{11}\mathbf{D} + \tilde{\mathbf{B}}_{12})(\tilde{\mathbf{B}}_{21}\mathbf{D} + \tilde{\mathbf{B}}_{22})^{-1}\hat{\mathbf{V}}_0^+, \\ \hat{\mathbf{V}}_0^- &= \mathbf{D}_{\text{eff}}\hat{\mathbf{V}}_0^+. \end{aligned} \quad (50)$$

where \mathbf{D}_{eff} is the effective reflection coefficient of the AL together with the boundary condition.

Note that even if the boundary condition is perfectly reflecting $\mathbf{D} = 0$, the AL can cause reflections, i.e.

$$\mathbf{D}_{\text{eff}} = \tilde{\mathbf{B}}_{12}\tilde{\mathbf{B}}_{22}^{-1} \neq 0, \quad (51)$$

unless $\tilde{\mathbf{B}}_{12} = 0$ or $\tilde{\mathbf{B}}_{22}$ is very large.

The best case is when $\tilde{\mathbf{B}}$ is block diagonal (i.e. $\tilde{\mathbf{B}}_{12} = \tilde{\mathbf{B}}_{21} = 0$). In that case we have

$$\begin{aligned} \hat{\mathbf{V}}_0^- &= \tilde{\mathbf{B}}_{11}\mathbf{D}\tilde{\mathbf{B}}_{22}^{-1}\hat{\mathbf{V}}_0^+, \\ \mathbf{D}_{\text{eff}} &= \tilde{\mathbf{B}}_{11}\mathbf{D}\tilde{\mathbf{B}}_{22}^{-1}. \end{aligned} \quad (52)$$

So, if in addition we can have that $\tilde{\mathbf{B}}_{11} \rightarrow 0$ and $\tilde{\mathbf{B}}_{22}^{-1} \rightarrow 0$, then the reflection will be negligible.

We can guarantee that $\tilde{\mathbf{B}}$ is block diagonal if we choose $\mathbf{C}^{-1}\mathbf{H}$ so that it is diagonal in the same basis as $\mathbf{C}^{-1}\mathbf{A}$. This can be done by looking at (35) and choosing \mathbf{H} of the form

$$\mathbf{H} = \mathbf{C}\mathbf{S}\mathbf{\Lambda}_H\mathbf{S}^{-1}, \quad (53)$$

where Λ_H is a diagonal matrix. We can show that $\tilde{\mathbf{B}}$ is also diagonal (see (44))

$$\begin{aligned}\mathbf{A}^{-1}(\mathrm{i}\omega\mathbf{C} + \mathbf{H})x &= (\mathbf{C}^{-1}\mathbf{A})^{-1}(\mathrm{i}\omega\mathbf{I} + \mathbf{C}^{-1}\mathbf{H}), \\ &= \mathbf{S}\Lambda^{-1}(\mathrm{i}\omega\mathbf{I} + \Lambda_H)\mathbf{S}^{-1}, \\ \mathbf{B} &= \mathbf{S} \exp\{\Lambda^{-1}(\mathrm{i}\omega\mathbf{I} + \Lambda_H)L\} \mathbf{S}^{-1},\end{aligned}\tag{54}$$

so that

$$\tilde{\mathbf{B}} = \exp\{\Lambda^{-1}(\mathrm{i}\omega\mathbf{I} + \Lambda_H)L\},\tag{55}$$

is diagonal, with diagonal elements

$$\tilde{B}_{jj} = \exp\left\{\frac{(\mathrm{i}\omega + \lambda_{H,j})L}{\lambda_j}\right\},\tag{56}$$

Moreover we note that

$$\begin{aligned}\tilde{\mathbf{B}}_{12} &= \tilde{\mathbf{B}}_{21} = 0, \\ \tilde{\mathbf{B}}_{11} &= \exp\{\Lambda_{11}^{-1}(\mathrm{i}\omega\mathbf{I} + \Lambda_{H,11})L\}, \\ \tilde{\mathbf{B}}_{22} &= \exp\{\Lambda_{22}^{-1}(\mathrm{i}\omega\mathbf{I} + \Lambda_{H,22})L\}.\end{aligned}\tag{57}$$

Now, recall that from the decomposition (38) we have that all the (diagonal) elements of Λ_{11} are positive, and those of Λ_{22} are negative. If we take Λ_H in such a way that all its (diagonal) elements are positive then

$$\tilde{\mathbf{B}}_{11} \rightarrow 0, \quad \text{and} \quad \tilde{\mathbf{B}}_{22} \rightarrow \infty, \quad \text{as} \quad L \rightarrow \infty.\tag{58}$$

as it was required for a full AL after (52).

Two possibilities are

- (AL1):

$$\Lambda_H \propto \mathbf{I}, \quad \implies \quad \mathbf{H} = K\mathbf{C}.\tag{59}$$

- (AL2):

$$\Lambda_H \propto |\Lambda|, \quad \implies \quad \mathbf{H} = K\mathbf{C}|\mathbf{C}^{-1}\mathbf{A}|.\tag{60}$$

where K is a user defined factor to control the intensity of the AL.

5.4 Choice of the absorption intensity factor K

Another advantage of using AL's based on (59,60) is that the equations are dimensionally correct, provided that K has dimensions of reciprocal of time for (AL1) and reciprocal of length for (AL2).

In the case of the second choice we can guarantee a certain damping rate per length of the AL. Note that

$$\begin{aligned}\tilde{\mathbf{B}}_{22} &= \exp\{\Lambda_{22}^{-1}(\mathrm{i}\omega\mathbf{I} + \Lambda_{H,22})L\}, \\ &= \text{diag}\{\exp\{\lambda_j^{-1}(\mathrm{i}\omega + K|\lambda_j|)L\}\}, \\ &= e^{KL} \text{diag}\{\exp\{\lambda_j^{-1}\mathrm{i}\omega L\}\}.\end{aligned}\tag{61}$$

So that $\|\tilde{\mathbf{B}}_{22}^{-1}\| \propto e^{-KL} \rightarrow 0$, provided that $KL \rightarrow \infty$. In the same way we can show that $\|\tilde{\mathbf{B}}_{11}\| \propto e^{-KL} \rightarrow 0$. So that both waves left-going and right-going have a uniform exponential decay with a rate proportional to e^{-KL} , and the composed effective reflection coefficient of the absorbing layer is

$$\|\mathbf{D}_{\text{eff}}\| \propto e^{-2KL}. \quad (62)$$

5.5 Absorption layer expressions for still water

Section §5.1 assumes that the state vector is represented as $\mathbf{U} = [h, \mathbf{u}]^T$, while section §5.5 assumes the converse, i.e. $\mathbf{U} = [\mathbf{u}, h]^T$ (which is the ordering used in PETSc-FEM).

The expressions of the involved matrices \mathbf{C} and \mathbf{A} for 1D shallow water are

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} h & u \\ 0 & 1 \end{bmatrix}, \\ \mathbf{A} &= \begin{bmatrix} 2hu & u^2 + gh \\ h & u \end{bmatrix}. \end{aligned} \quad (63)$$

If the reference state is still water, then $u = 0$ and they reduce to

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}, \\ \mathbf{A} &= \begin{bmatrix} 0 & gh \\ h & 0 \end{bmatrix}. \end{aligned} \quad (64)$$

So that the simplest expression for the absorbing layer (AL1) (see (59)) is

$$\mathbf{H} = K\mathbf{C} = K \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \quad (65)$$

where K is a positive number controlling the intensity of the AL. The second alternative can be computed using the eigenvalue decomposition given in (24). As the two eigenvalues are $\pm c$, with $c = \sqrt{gh}$, it turns out to be that $|\mathbf{\Lambda}| = \mathbf{I}$ and so

$$|\mathbf{C}^{-1}\mathbf{A}| = \mathbf{S}|\mathbf{\Lambda}|\mathbf{S}^{-1} = \mathbf{I}, \quad (66)$$

and then

$$\mathbf{H} = K\mathbf{C}|\mathbf{C}^{-1}\mathbf{A}| = K\mathbf{C}, \quad (67)$$

which is the same as (65), so that in this special case (still water), both criteria coincide in the same AL.

5.6 Absorbing layer based on the H=C operator (ALHC)

The condition for the matrix \mathbf{H} (see eq 25) to be an absorbing matrix is (see eq. 104) that $\mathbf{A}^{-1}\mathbf{G}$ must be diagonal in the basis of the eigenvectors of $\mathbf{A}^{-1}\mathbf{C}$ and

$$\text{sign}(\lambda(\mathbf{A}^{-1}\mathbf{H})) = \text{sign}(\lambda(\mathbf{A}^{-1}\mathbf{C})). \quad (68)$$

The simplest choice is $\mathbf{H} = \mathbf{C}$; we will call this the ALHC absorbing layer. Note that in this case waves traveling in both directions are damped, not only the incoming ones. This may be beneficial or not. If we can damp only the incoming, then the results in the absorbing layer are physically correct; the operator only damps the incoming waves, which are spurious. On the other hand, if we discard the values in the absorbing layer, then damping the waves in both directions may give a stronger damping for a layer of the same length.

The modified shallow water equations for the ALHC (i.e. the replacement for (27)) are simply

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + K(h - h_0) &= 0, \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{2}u^2 + gh) + K(u - u_0) &= 0. \end{aligned} \quad (69)$$

Note that now K has dimensions of s^{-1} . The absorbing term in the momentum equation can be implemented directly in the Navier-Stokes momentum equation. For the continuity equation it is not so direct. First note that the continuity equation for the SW comes from integrating the continuity equation in the vertical direction

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \frac{\partial w}{\partial z} + \nabla_{xy} \mathbf{v} = 0, \\ \int_{z=0}^{z=h} (\frac{\partial w}{\partial z} + \nabla_{xy} \mathbf{v}) &= 0, \\ w_{z=h} - w_{z=0} + \int_{z=0}^{z=h} \nabla_{xy} \cdot \mathbf{v} &= 0, \\ \frac{\partial h}{\partial t} + \nabla_{xy} \cdot (h\mathbf{v}) &= 0. \end{aligned} \quad (70)$$

where $\nabla_{xy} \cdot$ is the divergence operator in 2D, and $\mathbf{v} = (u, v)$ is the 2D projection on the xy -plane of the velocity vector. Note that if we want to get the absorbing term $K(h - h_0)$ in the SW equations, then we can add a term $K(h - h_0)/h_0$ in the NS continuity equation, so that after integrating over z we get the desired term.

In conclusion the NS-AL equations are

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\nabla p}{\rho} - \nu \Delta \mathbf{u} + K(\mathbf{u} - \mathbf{u}_0), \\ \nabla \cdot \mathbf{u} + K(h - h_0)/h_0 &= 0. \end{aligned} \quad (71)$$

5.7 Decaying rates for the ALHC

As a test case, if the water height is constant $h = h_1$ at $t = 0$, then the solution is $\mathbf{u} \equiv 0$, $h(t) = h_0 + (h_1 - h_0)e^{-Kt}$.

On the other hand, if the initial condition is

$$h = h_0 + A \cos(kx), \quad \mathbf{u} = 0, \quad (72)$$

with $A \ll h_0$ then the solution can be found as a superposition of plane waves. Note that in general, for the linear system with absorbing layer

$$\mathbf{C} \frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + K \mathbf{H} \mathbf{U} = 0, \quad (73)$$

if we multiply at left by \mathbf{C}^{-1} and assume that $\mathbf{A}^{-1}\mathbf{C}$ and $\mathbf{A}^{-1}\mathbf{H}$ are diagonal in the same basis

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{C} &= \mathbf{V} \Lambda_{\mathbf{C}} \mathbf{V}^{-1}, \\ \mathbf{A}^{-1}\mathbf{H} &= \mathbf{V} \Lambda_{\mathbf{H}} \mathbf{V}^{-1}, \end{aligned} \quad (74)$$

We find then plane wave solutions for the infinite case in the form of

$$\mathbf{U}(x, t) = \tilde{\mathbf{U}} e^{i(kx - \omega t)}. \quad (75)$$

Replacing in (73) we find

$$\begin{aligned} -i\omega \mathbf{C} + ik\mathbf{A} + \mathbf{H} \tilde{\mathbf{U}} &= 0, \\ \mathbf{A}(-i\omega \mathbf{A}^{-1}\mathbf{C} + ik\mathbf{I} + K\mathbf{A}^{-1}\mathbf{H})\tilde{\mathbf{U}} &= 0, \\ \mathbf{A}\mathbf{V}(-i\omega \Lambda_{\mathbf{C}} + ik\mathbf{I} + K\Lambda_{\mathbf{H}})\mathbf{V}^{-1}\tilde{\mathbf{U}} &= 0 \end{aligned} \quad (76)$$

o that (75) is a solution if $\tilde{\mathbf{U}}$ is an eigenvector of $\mathbf{A}^{-1}\mathbf{C}$, $\mathbf{A}^{-1}\mathbf{H}$, and

$$\omega = \frac{k - iK\lambda_{\mathbf{H}}}{\lambda_{\mathbf{C}}}. \quad (77)$$

In the case of of SW with the ALHC we have $\lambda_{\mathbf{H}} = \lambda_{\mathbf{C}} = \pm c_0$ so that

$$\omega = \pm \frac{k}{c_0} - iK, \quad (78)$$

so that the solution for a standing wave like (72) is a decaying rate

$$h = h_0 + A \cos(kx) \cos\left(\frac{kt}{c_0}\right) e^{-Kt}. \quad (79)$$

6 Absorption layers of higher order

The reflection coefficient of the layers described above is $O(\theta)$, where θ is the incidence angle (defined by $ky/kx = \tan \theta$). We try to develop in this section layers of higher order, i.e. that the reflection coefficient is $O(\theta^n)$ with $n > 1$. For this we need a better estimation of the incoming waves, and first we need an estimation of how the eigenvectors and eigenvalues of the Jacobians change for small angles; this is obtained using a perturbation theory.

6.1 First order perturbation theory for non symmetric operators

We assume that we want to solve

$$\mathbf{A}_\epsilon \mathbf{V}_\epsilon = \mathbf{V}_\epsilon \mathbf{\Lambda}_\epsilon, \quad (80)$$

where ϵ is a small perturbation parameter. We assume that this corresponds to the eigenvalue problem for the plane wave analysis of an hyperbolic system of PDE's so that $\mathbf{\Lambda}$ and \mathbf{V} are real. We want to obtain expressions for the change in the eigenvalues λ and the eigenvector \mathbf{V}_j (columns of \mathbf{V}) at first order in ϵ , so we assume a regular expansion

$$\begin{aligned} \mathbf{A}_\epsilon &= \mathbf{A} + \epsilon \delta \mathbf{A}, \\ \mathbf{V}_\epsilon &= \mathbf{V} + \epsilon \delta \mathbf{V}, \\ \mathbf{\Lambda}_\epsilon &= \mathbf{\Lambda} + \epsilon \delta \mathbf{\Lambda}, \end{aligned} \quad (81)$$

with $\delta \mathbf{V}$, $\delta \mathbf{\Lambda}$ real.

6.2 Eigenvalue perturbation

Replacing and retaining only the $O(\epsilon)$ terms we obtain

$$\begin{aligned} \delta \mathbf{A} \mathbf{V} + \mathbf{A} \delta \mathbf{V} &= \delta \mathbf{V} \mathbf{\Lambda} + \mathbf{V} \delta \mathbf{\Lambda}, \\ \mathbf{V}^{-1} \delta \mathbf{A} \mathbf{V} + \mathbf{V}^{-1} \mathbf{A} \delta \mathbf{V} &= \mathbf{V}^{-1} \delta \mathbf{V} \mathbf{\Lambda} + \delta \mathbf{\Lambda}, \end{aligned} \quad (82)$$

but

$$\mathbf{A} \mathbf{V} = \mathbf{V} \mathbf{\Lambda} \quad \implies \quad \mathbf{V}^{-1} \mathbf{A} = \mathbf{\Lambda} \mathbf{V}^{-1} \quad (83)$$

and replacing in (82)

$$\mathbf{V}^{-1} \delta \mathbf{A} \mathbf{V} = \mathbf{V}^{-1} \delta \mathbf{V} \mathbf{\Lambda} - \mathbf{\Lambda} \mathbf{V}^{-1} \delta \mathbf{V} + \delta \mathbf{\Lambda}. \quad (84)$$

Now note that the diagonal elements of the first two terms cancel because $\mathbf{\Lambda}$ is diagonal so that

$$(\mathbf{V}^{-1} \delta \mathbf{V} \mathbf{\Lambda})_{jj} = (\mathbf{V}^{-1} \delta \mathbf{V})_{jj} (\mathbf{\Lambda})_{jj} = (\mathbf{\Lambda} \mathbf{V}^{-1} \delta \mathbf{V})_{jj} \quad (85)$$

so that we can compute to first order in ϵ the perturbations in the eigenvalues by taking the diagonal part of (84)

$$\text{diag}\{\delta \mathbf{\Lambda}\} = \text{diag}\{\mathbf{V}^{-1} \delta \mathbf{A} \mathbf{V}\}. \quad (86)$$

6.3 Eigenvector perturbation, symmetric case.

Now assume that the system is symmetric, i.e. \mathbf{A}_ϵ is symmetric for all ϵ , so that \mathbf{A} , $\delta\mathbf{A}$ are symmetric. Then we can chose \mathbf{V} such that it is orthogonal, i.e. $\mathbf{V}^T\mathbf{V} = \mathbf{I}$, and then

$$\delta(\mathbf{V}^T\mathbf{V}) = \delta\mathbf{V}^T\mathbf{V} + \mathbf{V}^T\delta\mathbf{V} = 0, \quad (87)$$

but

$$(\mathbf{V}^T\delta\mathbf{V})^T = \delta\mathbf{V}^T\mathbf{V}, \quad (88)$$

so that $\mathbf{V}^T\delta\mathbf{V}$ is antisymmetric, so a typical off-diagonal element of (84) gives its elements

$$(\mathbf{V}^T\delta\mathbf{V})_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{(\mathbf{V}^T\delta\mathbf{A}\mathbf{V})_{ij}}{\lambda_j - \lambda_i}, & \text{if } i \neq j. \end{cases} \quad (89)$$

Note that the $\mathbf{V}^T\delta\mathbf{A}\mathbf{V}$ is symmetric, but eigenvalue difference in the denominator makes the quotient antisymmetric, as it should be. The expression is singular if there are multiple eigenvalues, which we will assume that does not happen. Once $\mathbf{V}^T\delta\mathbf{V}$ is computed with (89), $\delta\mathbf{V}$ is computed trivially by premultiplying by \mathbf{V}

$$\delta\mathbf{V} = \mathbf{V}(\mathbf{V}^T\delta\mathbf{V}). \quad (90)$$

6.4 Eigenvector perturbation, symmetrizable case.

The result can be extended to the more general case where the hyperbolic system is symmetrizable, i.e. there exists a transformation matrix \mathbf{S} such that $\mathbf{S}^{-1}\mathbf{A}_\epsilon\mathbf{S}$ for all ϵ . Note that the matrix \mathbf{S} must not depend on ϵ . If this is the case, we have to first order that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ and $\mathbf{S}^{-1}\delta\mathbf{A}\mathbf{S}$ must be symmetric (again, for the *same* \mathbf{S}). Let \mathbf{V}' , be the eigenvector basis for $\mathbf{A}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, i.e.

$$\mathbf{V}'^{-1}\mathbf{A}'\mathbf{V}' = \mathbf{V}'^{-1}\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{V}' = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}, \quad (91)$$

so that

$$\mathbf{V} = \mathbf{S}\mathbf{V}'. \quad (92)$$

Then applying the perturbation for symmetric systems of the previous section

$$\begin{aligned} \text{diag}\{\delta\mathbf{A}\} &= \text{diag}\{\mathbf{V}'^{-1}\delta\mathbf{A}'\mathbf{V}'\}, \\ &= \text{diag}\{(\mathbf{S}^{-1}\mathbf{V})^{-1}\delta\mathbf{A}'\mathbf{S}^{-1}\mathbf{V}'\}, \\ &= \text{diag}\{\mathbf{V}^{-1}\delta\mathbf{A}\mathbf{V}\}, \end{aligned} \quad (93)$$

so that the expression for the eigenvalue perturbation is almost the same as for the symmetric case, however note that in (86) we can use both \mathbf{V}^T or \mathbf{V}^{-1} because the transformation matrix is orthogonal, but in the non-symmetric case this is not true, so that the inverse must be used.

Similarly, regarding the eigenvectors perturbation we can apply (89) for the symmetric system

$$(\mathbf{V}'^T\delta\mathbf{V}')_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{(\mathbf{V}'^T\delta\mathbf{A}'\mathbf{V}')_{ij}}{\lambda_j - \lambda_i}, & \text{if } i \neq j. \end{cases} \quad (94)$$

Now coming back to the non-symmetric basis, we note that

$$\mathbf{V}'^T \delta \mathbf{V}' = \mathbf{V}'^{-1} \delta \mathbf{V}' = \mathbf{S}^{-1} \mathbf{V}^{-1} \delta (\mathbf{S}^{-1} \mathbf{V}) = \mathbf{V}^{-1} \mathbf{S} \mathbf{S}^{-1} \delta \mathbf{V} = \mathbf{V}^{-1} \delta \mathbf{V}, \quad (95)$$

and

$$\begin{aligned} \mathbf{V}'^T \delta \mathbf{A}' \mathbf{V}' &= (\mathbf{S}^{-1} \mathbf{V})^{-1} \delta \mathbf{A}' \mathbf{S}^{-1} \mathbf{V}, \\ &= \mathbf{V}^{-1} \mathbf{S} \delta \mathbf{A}' \mathbf{S}^{-1} \mathbf{V}, \\ &= \mathbf{V}^{-1} \delta \mathbf{A} \mathbf{V}, \end{aligned} \quad (96)$$

so that, again, the expression is almost the same as for the symmetric case

$$(\mathbf{V}^{-1} \delta \mathbf{V})_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{(\mathbf{V}^{-1} \delta \mathbf{A} \mathbf{V})_{ij}}{\lambda_j - \lambda_i}, & \text{if } i \neq j. \end{cases} \quad (97)$$

6.5 Design of the absorbing layer

Assuming that we want to add an absorbing layer in $0 \leq x \leq L$, then the equation would be

$$\mathbf{C} \frac{\partial \mathbf{U}}{\partial t} + \mathbf{H} \mathbf{U} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial y} = 0, \quad (98)$$

where \mathbf{H} is the matrix of absorbing coefficients to be yet defined. We now transform Fourier in t and y with associated variables $i\omega$ and ik_y , i.e. we assume

$$\mathbf{U}(x, y, t) = \hat{\mathbf{U}}(x) \exp\{i(k_y y - \omega t)\}. \quad (99)$$

Then, we get (for simplicity we drop the hat symbol ($\hat{\mathbf{U}} \rightarrow \mathbf{U}$),

$$\begin{aligned} \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + (-i\omega \mathbf{C} + \mathbf{H} + ik_y \mathbf{B}) \mathbf{U} &= 0, \\ \implies \frac{\partial \mathbf{U}}{\partial x} + (\mathbf{A}^{-1} \mathbf{H} - i\omega \mathbf{M}(z)) \mathbf{U} &= 0, \end{aligned} \quad (100)$$

where

$$\begin{aligned} z &= k_y / \omega, \\ \mathbf{M}(z) &= -\mathbf{A}^{-1} (\mathbf{C} - z \mathbf{B}). \end{aligned} \quad (101)$$

In order to not have reflections $\mathbf{A}^{-1} \mathbf{H}$ must have the same eigenvectors that $\mathbf{M}(z)$. One possibility is then to diagonalize $\mathbf{M}(z)$

$$\mathbf{M}(z) = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1}, \quad (102)$$

and then to force \mathbf{H} to be diagonal in the same basis

$$\mathbf{H} = \mathbf{A} \mathbf{Q} \mathbf{\Lambda}_H \mathbf{Q}^{-1}. \quad (103)$$

In order to have an absorbing layer we must have

$$\text{sign}(\lambda_{Hj}) = \text{sign}(\lambda_j), \quad (104)$$

Transforming Fourier back to t and y we obtain the desired absorbing layer. However, \mathbf{Q} depends on z and so will \mathbf{H} so it would be non-local. In order to have a local operator we perform an expansion of it in powers of z . For instance if we can approximate it to

$$\mathbf{H}(z) \approx \mathbf{H}_0 + z\delta\mathbf{H}, \quad (105)$$

then the absorbing term would be

$$\mathbf{H}_0\mathbf{U} + z\delta\mathbf{H}\mathbf{U} = \mathbf{H}_0\mathbf{U} + (ik_y/i\omega)\delta\mathbf{H}\mathbf{U}. \quad (106)$$

Transforming Fourier back to (t, y) we get

$$\begin{aligned} \mathbf{C}\frac{\partial\mathbf{U}}{\partial t} + \mathbf{H}(\{\mathbf{U}\}) + \mathbf{A}\frac{\partial\mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial\mathbf{U}}{\partial y} &= 0, \\ \mathbf{H}(\{\mathbf{U}\}) &= \mathbf{H}_0\mathbf{U} + \delta\mathbf{H}\int_{t=0}^t \frac{\partial\mathbf{U}}{\partial y} dt. \end{aligned} \quad (107)$$

6.6 Computation of the absorbing coefficient matrix

For $z = 0$ we have $\mathbf{M}(z) = -\mathbf{A}^{-1}$, and let \mathbf{Q}_0 be the matrix of eigenvectors for \mathbf{A} , so that

$$\mathbf{A} = \mathbf{Q}_0\mathbf{\Lambda}\mathbf{Q}_0^{-1}. \quad (108)$$

Note that the eigenvectors for $\mathbf{M}_{(z=0)} = -\mathbf{A}^{-1}$ will be the same as for \mathbf{A} but with eigenvalues $-1/\lambda_j$, where λ_j are the eigenvalues of \mathbf{A} . We can perform a perturbation analysis as in sections §6.3 and §6.4 where z is the perturbation parameter, and

$$\mathbf{M}(z) \approx \mathbf{M}_0 + z\delta\mathbf{M} = -\mathbf{A}^{-1} + z(-\mathbf{A}^{-1}\mathbf{B}), \quad (109)$$

and compute the eigenvectors perturbation $\delta\mathbf{Q}$ such that

$$\mathbf{Q}(z) \approx \mathbf{Q}_0 + z\delta\mathbf{Q}. \quad (110)$$

The absorbing operator can be computed to first order as

$$\mathbf{H}(z) \approx (\mathbf{Q}_0 + z\delta\mathbf{Q})\mathbf{\Lambda}_H(\mathbf{Q}_0^{-1} + z\delta(\mathbf{Q}^{-1})). \quad (111)$$

The perturbation in \mathbf{Q}^{-1} can be computed by differentiation of the identity $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{I}$, so that

$$\begin{aligned} \delta\mathbf{Q}\mathbf{Q}_0^{-1} + \mathbf{Q}_0\delta(\mathbf{Q}^{-1}) &= 0, \\ \implies \delta(\mathbf{Q}^{-1}) &= -\mathbf{Q}_0^{-1}\delta\mathbf{Q}\mathbf{Q}_0^{-1}. \end{aligned} \quad (112)$$

Then

$$\mathbf{H} = \mathbf{A}(\mathbf{Q}_0\mathbf{\Lambda}_H\mathbf{Q}_0^{-1} + z[\delta\mathbf{Q}\mathbf{\Lambda}_H\mathbf{Q}_0^{-1} + \mathbf{Q}_0\mathbf{\Lambda}_H\delta(\mathbf{Q}^{-1})]), \quad (113)$$

so that

$$\begin{aligned} \mathbf{H}_0 &= \mathbf{A}\mathbf{Q}_0\mathbf{\Lambda}_H\mathbf{Q}_0^{-1} = \mathbf{A}\mathbf{D}, \\ \delta\mathbf{H} &= \mathbf{A}[\delta\mathbf{Q}\mathbf{\Lambda}_H\mathbf{Q}_0^{-1} + \mathbf{Q}_0\mathbf{\Lambda}_H\delta(\mathbf{Q}^{-1})], \\ &= \mathbf{A}[\delta\mathbf{Q}\mathbf{\Lambda}_H\mathbf{Q}_0^{-1} - \mathbf{Q}_0\mathbf{\Lambda}_H\mathbf{Q}_0^{-1}\delta\mathbf{Q}\mathbf{Q}_0^{-1}], \\ &= \mathbf{A}[(\delta\mathbf{Q}\mathbf{Q}_0^{-1})\mathbf{D} - \mathbf{D}(\delta\mathbf{Q}\mathbf{Q}_0^{-1})], \end{aligned} \quad (114)$$

Note that, of course, if we choose $\mathbf{\Lambda}_H = K \text{sign}(\mathbf{\Lambda})$ then $\mathbf{H}_0 = K|A|$ (coincident with (60), here we assume $\mathbf{C} = \mathbf{I}$). On the other hand if we choose $\mathbf{\Lambda}_H = K\mathbf{\Lambda}^{-1}$ then $\mathbf{H}_0 = K\mathbf{I}$ (coincident with (59)).

6.7 Extension to 3D

Extension to 3D is straightforward (101) has now two perturbation parameters,

$$\begin{aligned}\eta &= k_y/\omega, \\ \zeta &= k_z/\omega, \\ \mathbf{M}(\eta, \zeta) &= -\mathbf{A}^{-1}(\mathbf{C} - \eta\mathbf{B} - \zeta\mathbf{C}).\end{aligned}\tag{115}$$

and after the perturbation analysis we get

$$\mathbf{H}(\eta, \zeta) \approx \mathbf{H}_0 + \eta\delta\mathbf{H}_y + \zeta\delta\mathbf{H}_z,\tag{116}$$

where both $\delta\mathbf{H}_{y,z}$ are obtained with a perturbation analysis as before, and then the absorbing operator is

$$\mathbf{H}(\{\mathbf{U}\}) = \mathbf{H}_0\mathbf{U} + \delta\mathbf{H}_y \int_{t=0}^t \frac{\partial\mathbf{U}}{\partial y} dt + \delta\mathbf{H}_z \int_{t=0}^t \frac{\partial\mathbf{U}}{\partial z} dt.\tag{117}$$

6.8 Numerical evaluation of the absorbing operator

- It involves an integral in time, that involves storing an auxiliary variable say \mathbf{W} and then updating with

$$\frac{(\mathbf{W}^{n+1} - \mathbf{W}^n)_{jk}}{\Delta t} = \frac{(\mathbf{U}_{j,k+1} - \mathbf{U}_{j,k-1})^{n+\theta}}{2\Delta y}.\tag{118}$$

$0 \leq \theta \leq 1$ is a trapezoidal rule parameter.

- The derivative with respect to y is computed by standard finite difference approximations ($j(k)$ indices is along $x(y)$ axis).
- It seems that the operator could be easily evaluated in the context of an unstructured grid solver (FEM for instance).

6.8.1 Second order precision in time implicit

Another possibility second order precise at t^{n+1} is

$$\frac{(3\mathbf{W}^{n+1} - 4\mathbf{W}^n + \mathbf{W}^{n-1})_{jk}}{2\Delta t} = \frac{(\mathbf{U}_{j,k+1} - \mathbf{U}_{j,k-1})^{n+1}}{2\Delta y}.\tag{119}$$

6.8.2 Relaxation time

A problem with (107) is that if \mathbf{W} doesn't loose track of the history of \mathbf{U} , i.e. if some error is performed during the integration, then this error persists forever. We then introduce a relaxation time τ and modify the relation between \mathbf{W} and \mathbf{U} in the following way

$$\dot{\mathbf{W}} + \frac{1}{\tau}\mathbf{W} = \frac{\partial\mathbf{U}}{\partial y}.\tag{120}$$

The corresponding numerical approximation is

$$\frac{(3f\mathbf{W}^{n+1} - 4\mathbf{W}^n + \mathbf{W}^{n-1})_{jk}}{2\Delta t} = \frac{(\mathbf{U}_{j,k+1} - \mathbf{U}_{j,k-1})^{n+1}}{2\Delta y},$$

$$f = 1 + \frac{2\Delta t}{3\tau}.$$
(121)

6.8.3 Second order spatial operator

If we consider a higher (second order) approximation for $\mathbf{H}(z)$

$$\mathbf{H}(z) \approx \mathbf{H}_0 + z\mathbf{H}_1 + z^2\mathbf{H}_2,$$
(122)

then the absorbing term would be

$$\mathbf{H}_0\mathbf{U} + z\mathbf{H}_1\mathbf{U} + z^2\mathbf{H}_2\mathbf{U} = \mathbf{H}_0\mathbf{U} + (ik_y/i\omega)\mathbf{H}_1\mathbf{U} + (ik_y/i\omega)^2\mathbf{H}_2\mathbf{U}.$$
(123)

Transforming Fourier back to (t, y) we get

$$\mathbf{C}\frac{\partial\mathbf{U}}{\partial t} + \mathbf{H}(\{\mathbf{U}\}) + \mathbf{A}\frac{\partial\mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial\mathbf{U}}{\partial y} = 0,$$

$$\mathbf{H}(\{\mathbf{U}\}) = \mathbf{H}_0\mathbf{U} + \mathbf{H}_1\mathbf{W}_1 + \mathbf{H}_2\mathbf{W}_2,$$
(124)

where in general \mathbf{W}_m satisfies,

$$\frac{\partial^n \mathbf{W}_m}{\partial t^m} = \frac{\partial^m \mathbf{U}}{\partial y^m}.$$
(125)

This can be computed numerically for instance as follows

$$\frac{(2\mathbf{W}^{n+1} - 5\mathbf{W}^n + 4\mathbf{W}^{n-1} - \mathbf{W}^{n-2})_{2,jk}}{\Delta t^2} = \frac{(\mathbf{U}_{j,k+1} - 2\mathbf{U}_{j,k} + \mathbf{U}_{j,k-1})^{n+1}}{\Delta y^2}.$$
(126)

So that the the second term in (124) can be computed as

$$(\mathbf{H}_2\mathbf{W}_2)_{jk}^{n+1} \approx \frac{1}{2}\mathbf{H}_2 \left[\frac{\Delta t^2}{\Delta y^2} (\mathbf{U}_{j,k+1} - 2\mathbf{U}_{j,k} + \mathbf{U}_{j,k-1})^{n+1} - \bar{\mathbf{W}} \right]$$

$$\bar{\mathbf{W}} = (-5\mathbf{W}^n + 4\mathbf{W}^{n-1} - \mathbf{W}^{n-2})_{2,jk}$$
(127)

Note that the $\bar{\mathbf{W}}$ does not depend on the values of \mathbf{U} at instant $n + 1$, which are the unknowns of the problem to be determined in this time step.



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