Dynamic Absorbing Boundary Conditions for Advective-Difusive Systems with Unknown Riemann Invariants

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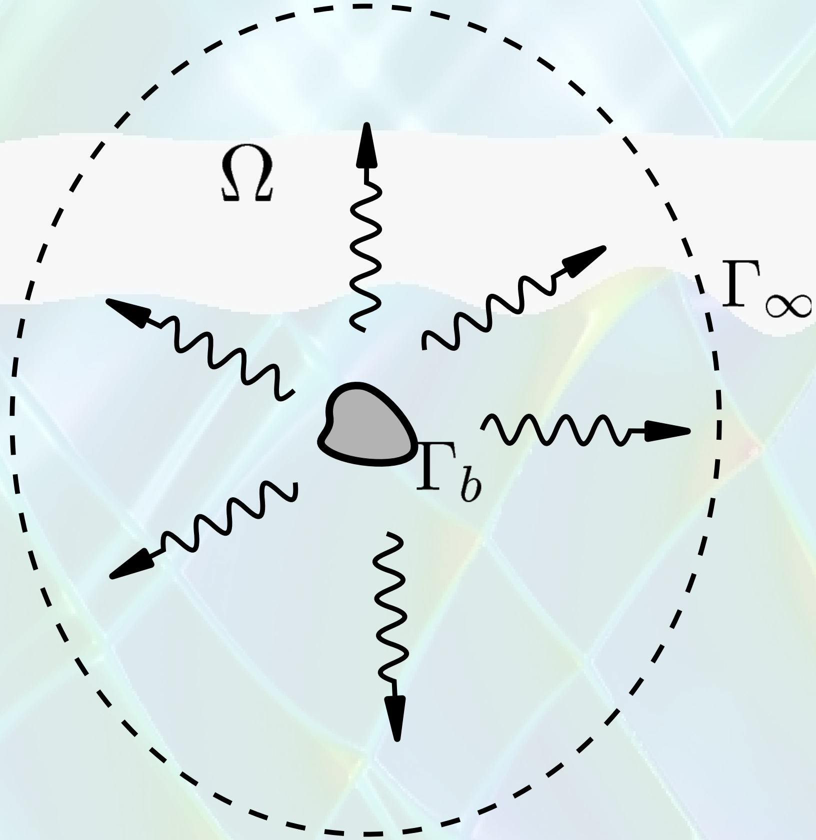
Motivation for Absorbing Boundary Conditions

In wave-like propagation problems, not including ABC may lead to non-convergent solutions.

$$u_{tt} = c^2 \Delta u, \text{ in } \Omega$$

$$u = \bar{u}(x,t), \text{ at } \Gamma_b, \quad \boxed{u = 0, \text{ at } \Gamma_\infty v}$$

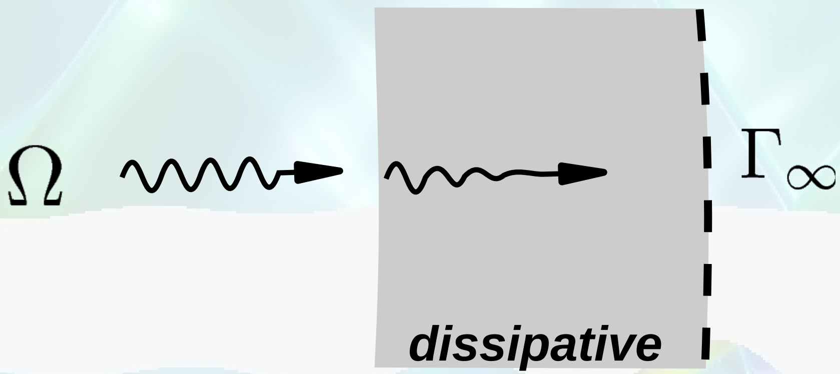
- u **doesn't converge to the correct solution irradiating energy from the source, even if $\Gamma_\infty \to \infty$. A *standing wave* is always found.**
- u **is unbounded if $\bar{u}$ emits in an eigenfrequency which is a resonance mode of the closed cavity.**



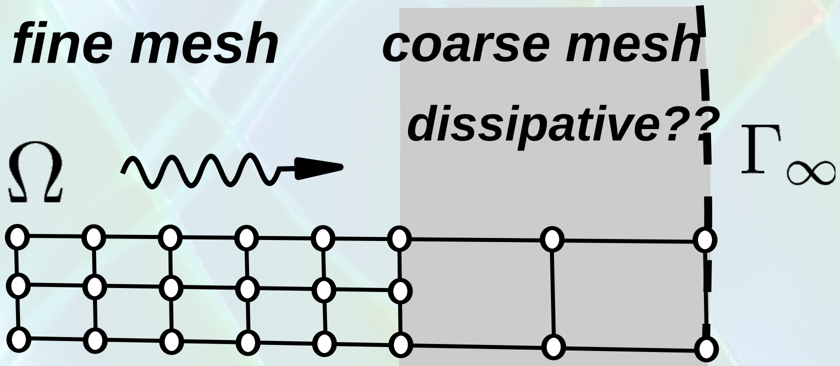
Absorbing boundary conditions must be added to the outer boundary in order to let energy be extracted from the domain.

Motivation for Absorbing Boundary Conditions (cont.)

If a dissipative region is large enough so that many wavelengths are included in the region may act as an absorbing layer, i.e. as an absorbing boundary condition.



However, it is a common misconception to assume that a coarse mesh adds dissipation and consequently may improve absorption. For instance for the wave equation a coarse mesh may lead to evanescent solutions and then to act as a fully reflecting boundary.



Boundary conditions for advective diffusive systems

Well known theory and practice for advective systems say that at a boundary the number of Dirichlet conditions should be equal to the number of incoming characteristics.

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathcal{F}_{c,j}(\mathbf{U})}{\partial x_j} = 0$$

$$A_{c,j} = \frac{\partial \mathcal{F}_{c,j}(\mathbf{U})}{\partial \mathbf{U}},$$ **advective Jacobian**

Nbr. of incoming characteristics $= \boxed{\mathrm{sum}(\mathrm{eig}(\mathbf{A}\cdot\hat{\mathbf{n}}) < 0)}$

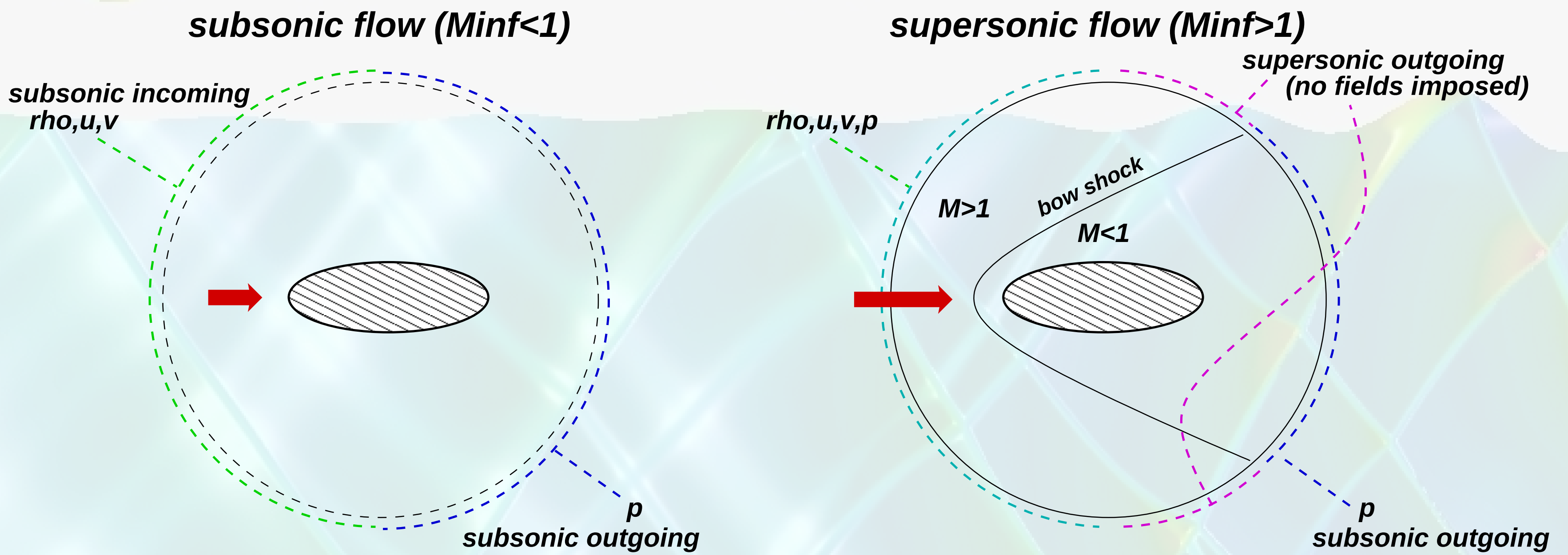
$\hat{\mathrm{n}}$ **is the exterior normal.**

Adding extra Dirichlet conditions leads to spurious shocks, and lack of enough Dirichlet conditions leads to instability.

Boundary conditions for advective diffusive systems (cont.)

For simple scalar advection problems the Jacobian is the transport velocity. The rule is then to check the projection of velocity onto the exterior normal.

For more complex flows (i.e. with **non diagonalizable Jacobians**, as gas dynamics or shallow water eqs.) the number of incoming characteristics may be approx. predicted from the flow conditions.



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Absorbing boundary conditions

However, this kind of conditions are, generally, reflective. Consider a pure advective system of equations in 1D, i.e., $\mathcal{F}_{d,j} \equiv 0$

$$\frac{\partial \mathcal{H}(\mathbf{U})}{\partial t} + \frac{\partial \mathcal{F}_{c,x}(\mathbf{U})}{\partial x} = 0, \text{ in } [0, L]. \tag{1}$$

If the system is *"linear"*, i.e., $\mathcal{F}_{c,x}(\mathbf{U}) = \mathbf{A}\mathbf{U}$, $\mathcal{H}(\mathbf{U}) = \mathbf{C}\mathbf{U}$ ($\mathbf{A}$ and $\mathbf{C}$ do not depend on $\mathbf{U}$), a first order linear system is obtained

$$\mathbf{C}\,\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\,\frac{\partial \mathbf{U}}{\partial x} = 0. \tag{2}$$

The system is *"hyperbolic"* if $\mathbf{C}$ is invertible, $\mathbf{C}^{-1}\mathbf{A}$ is diagonalizable with real eigenvalues. If this is the case, it is possible to make the following eigenvalue decomposition for $\mathbf{C}^{-1}\mathbf{A}$

$$\mathbf{C}^{-1}\mathbf{A} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1}, \tag{3}$$

where $\mathbf{S}$ is real and invertible and $\boldsymbol{\Lambda}$ is real and diagonal. If new variables are

defined $\mathbf{V} = \mathbf{S}^{-1}\mathbf{U}$, then equation (2) becomes

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{\Lambda} \frac{\partial \mathbf{V}}{\partial x} = 0. \tag{4}$$

Now, each equation is a linear scalar advection equation

$$\frac{\partial v_k}{\partial t} + \lambda_k \frac{\partial v_k}{\partial x} = 0, \quad \textbf{(no summation over } k\textbf{).} \tag{5}$$

v_k **are the *"characteristic components"* and λ_k are the *"characteristic velocities"* of propagation.**

Linear 1D absorbing boundary conditions

Assuming $\lambda_k \neq 0$, the absorbing boundary conditions are, depending on the sign of λ_k,

$$\begin{aligned} &\textbf{if } \lambda_k > 0\textbf{:}\ v_k(0) = \bar{v}_{k0}; \quad \textbf{no boundary condition at } x = L \\ &\textbf{if } \lambda_k < 0\textbf{:}\ v_k(L) = \bar{v}_{kL}; \quad \textbf{no boundary condition at } x = 0 \end{aligned} \tag{6}$$

This can be put in compact form as

$$\begin{aligned} \mathbf{\Pi}_V^+(\mathbf{V} - \bar{\mathbf{V}}_0) = 0; &\qquad \textbf{at } x = 0 \\ \mathbf{\Pi}_V^-(\mathbf{V} - \bar{\mathbf{V}}_L) = 0; &\qquad \textbf{at } x = L \end{aligned} \tag{7}$$

Linear 1D absorbing boundary conditions (cont.)

$\mathbf{\Pi}_V^\pm$ are the *projection matrices onto the right/left-going characteristic modes* in the $\mathbf{V}$ basis,

$$\mathbf{\Pi}^+_{V,jk} = \begin{cases} 1; & \text{if } j = k \text{ and } \lambda_k > 0 \\ 0; & \text{otherwise}, \end{cases} \tag{8}$$

$$\mathbf{\Pi}^+ + \mathbf{\Pi}^- = \mathbf{I}.$$

It can be easily shown that they are effectively *projection matrices*, i.e., $\mathbf{\Pi}^\pm\mathbf{\Pi}^\pm = \mathbf{\Pi}^\pm$ and $\mathbf{\Pi}^+\mathbf{\Pi}^- = 0$. Coming back to the boundary condition at $x = L$ in the $\mathbf{U}$ basis, it can be written

$$\mathbf{\Pi}_V^- \, \mathbf{S}^{-1}(\mathbf{U} - \bar{\mathbf{U}}_L) = 0 \tag{9}$$

or, multiplying by $\mathbf{S}$ at the left

$$\mathbf{\Pi}_U^\pm \, (\mathbf{U} - \bar{\mathbf{U}}_{0,L}) = 0, \quad \text{at } x = 0, L, \tag{10}$$

where

$$\mathbf{\Pi}_U^\pm = \mathbf{S}\, \mathbf{\Pi}_V^\pm \, \mathbf{S}^{-1}, \tag{11}$$

Linear 1D absorbing boundary conditions (cont.)

$$\begin{aligned} &\mathbf{\Pi}_U^{\pm}\,(\mathbf{U} - \bar{\mathbf{U}}_{0,L}) = 0, \quad \text{at } x = 0, L, \\ &\mathbf{\Pi}_U^{\pm} = \mathbf{S}\,\mathbf{\Pi}_V^{\pm}\,\mathbf{S}^{-1}, \end{aligned} \tag{12}$$

These conditions are completely absorbing for 1D linear advection system of equations (2).

The rank of $\mathbf{\Pi}^+$ is equal to the number n_+ of positive eigenvalues, i.e., the number of right-going waves. Recall that the right-going waves are incoming at the $x = 0$ boundary and outgoing at the $x = L$ boundary. Conversely, the rank of $\mathbf{\Pi}^-$ is equal to the number n_- of negative eigenvalues, i.e., the number of left-going waves (incoming at $x = L$ and outgoing at the $x = 0$ boundary).

ABC for nonlinear problems

First order absorbing boundary conditions may be constructed by imposing exactly the components along the incoming characteristics.

$$\mathbf{\Pi}^{-}(\mathbf{U}_{\mathrm{ref}})\,(\mathbf{U}-\mathbf{U}_{\mathrm{ref}})=0.$$

$\mathbf{\Pi}^{-}$ is the projection operator onto incoming characteristics. It can be obtained straightforwardly from the projected Jacobian.

This assumes linearization of the equations around a state $\mathrm{U}_{\mathrm{ref}}$. For linear problems $\mathrm{A}_{c,j}$ do not depend on U, and then neither the projection operator, so that absorbing boundary conditions coefficients are constant.

ABC for nonlinear problems (cont.)

For non-linear problems the Jacobian and projection operator may vary and then the above mentioned b.c.'s are not fully absorbing.

In some cases the concept of characteristic component may be extended to the non-linear case: the "Riemann invariants". Fully absorbing boundary conditions could be written in terms of the invariants:

$$w_j = w_{\mathrm{ref},j}, \quad \text{if } w_j \text{ is an incoming R.I.}$$

- **R.I. are computed analytically. There are no automatic (numerical) techniques to compute them. (They amount to compute an integral in phase space along a specific path).**
- **R.I. are known for shallow water, channel flow (for rectangular $w_j = \mathbf{u}\cdot\hat{\mathrm{n}} \pm 2\sqrt{gh}$, and triangular channel shape $w_j = \mathbf{u}\cdot\hat{\mathrm{n}} \pm 4\sqrt{gh}$). For gas dynamics the well known R.I. in fact are invariants only under isentropic conditions (i.e. not truly invariant).**

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ABC for nonlinear problems (cont.)

Search for an absorbing boundary condition that

- **should be fully absorbent in non-linear conditions, and**
- **can be computed numerically (no need of analytic expressions like R.I.)**

Solution: **Use last state as reference state, ULSAR**.

$\mathbf{U}_{\rm ref} = \mathbf{U}^n$, $n =$ **time step number.**

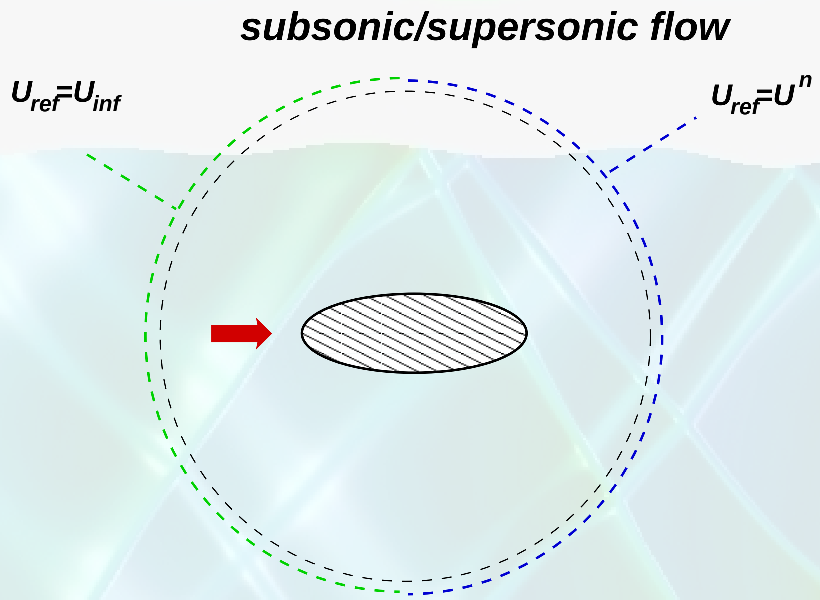
$$\mathbf{\Pi}^- (\mathbf{U}^n) \left(\mathbf{U}^{n+1} - \mathbf{U}^n\right) = 0.$$

As $\mathbf{U}^{n+1} - \mathbf{U}^n$ **is usually small, linearization is valid.**

ABC for nonlinear problems (cont.)

Disadvantage: **Flow conditions are only determined from the initial state. No external information comes from the outside.**

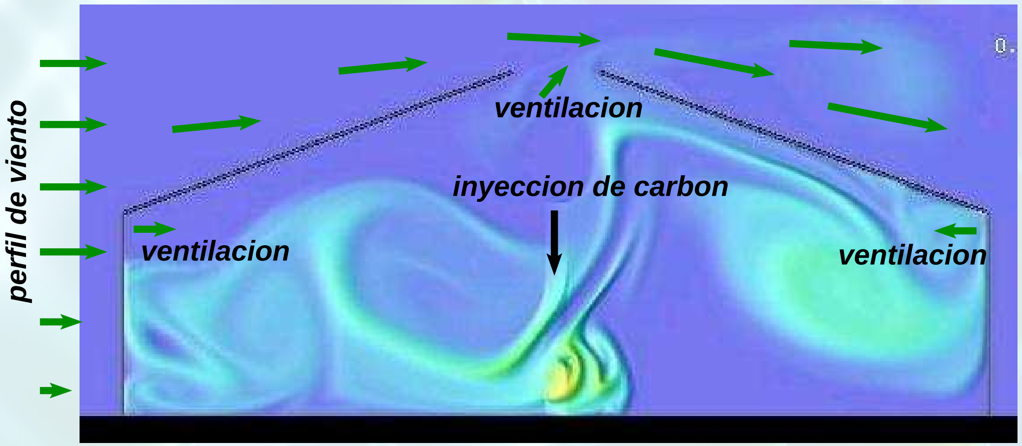
Solution: use a combination of linear/R.I. b.c.'s on incoming boundaries and use fully non-linear a.b.c's with previous state as reference state at the outlet.



Dynamic boundary conditions

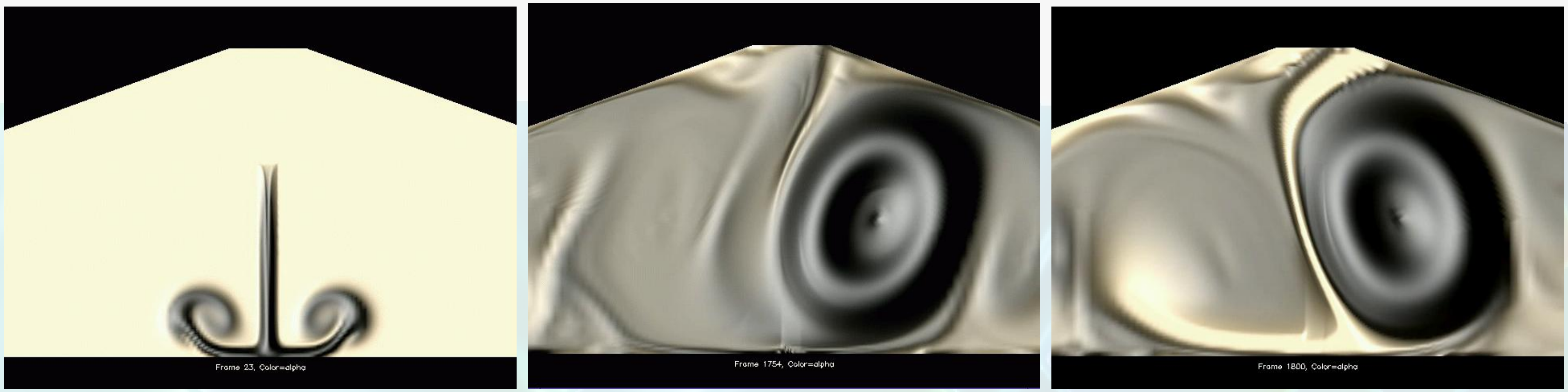
As the flow is computed it may happen that the number of characteristics changes in time. Two examples follow.

Think at transport of a scalar on top of a velocity field obtained by an incompressible Navier-Stokes solver (not truly that in the example since both systems are coupled). If the exterior flow is not modeled then flow at the top opening may be reverted.



Dynamic boundary conditions (cont.)

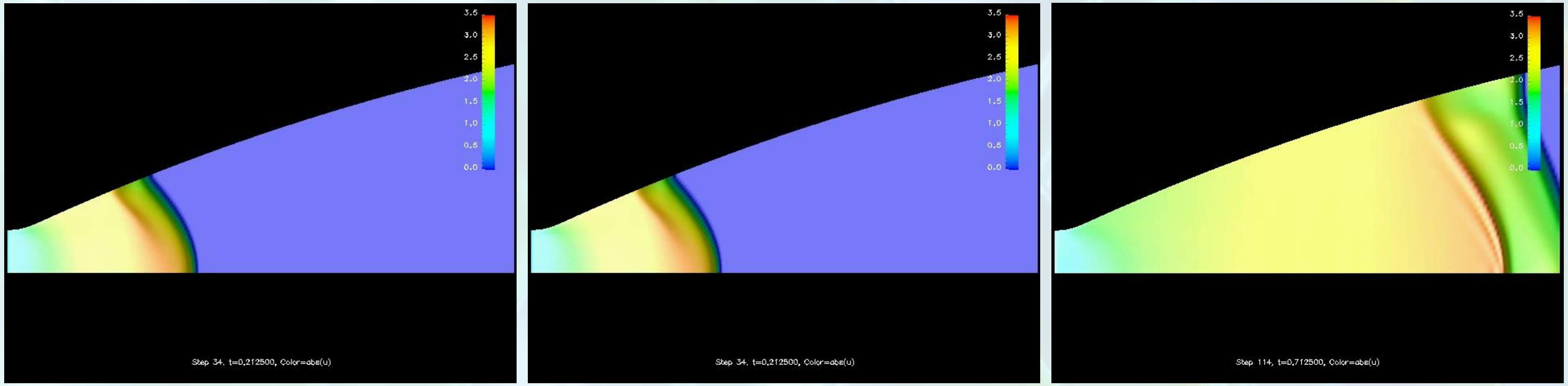
If the interior is modeled only, then it's natural to leave concentration free at the top opening. However the flow can revert at some portions of the opening producing a large incoming of undetermined values (in practice large negative concentrations are observed). Imposing a value at the opening is stable but would lead to a spurious discontinuity at the outlet.



The ideal would be to switch dynamically from one condition to the other during the computation.

Nozzle chamber fill

The case is the ignition of a rocket launcher nozzle in a low pressure atmosphere. The fluid is initially at rest (143 Pa, 262 $^\circ\mathrm{K}$). At time $t = 0$ a membrane at the throat is broken. Behind the membrane there is a reservoir at 6×10^5 Pa, $4170^\circ\mathrm{K}$. A strong shock (intensity $p_1/p_2 >$1000) propagates from the throat to the outlet. The gas is assumed as ideal ($\gamma = 1.17$). In the steady state a supersonic flow with a max. Mach of 4 at the outlet is found. The objective of the simulation is to determine the time needed to fill the chamber (< 1msec) and the final steady flow.



Nozzle chamber fill (cont.)

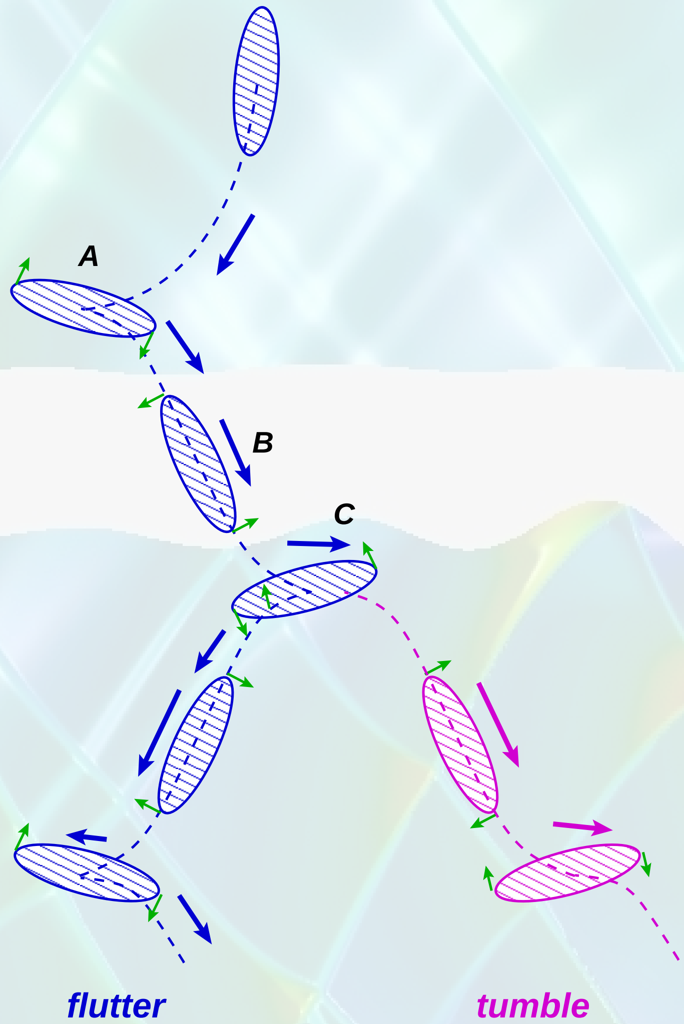
We impose density, pressure and tangential velocity at inlet (assuming subsonic inlet), slip condition at the nozzle wall. The problem is with the outlet boundary. Initially the flow is subsonic (fluid at rest) there, and switches to supersonic. The rule dictaminates to impose 1 condition, as a subsonic outlet (may be pressure, which is known) and no conditions after (supersonic outlet). If pressure is imposed during the wall computation, then a spurious shock is formed at the outlet.

This test case has been contrasted with experimental data obtained at ESTEC/ESA (European Space Research and Technology Centre-European Space Agency, Noordwijk, Holanda). The predicted mean velocity was $2621\ \mathrm{m/s}$ to be compared with the experimental value of $2650 \pm 50\ \mathrm{m/sec}$.

Again, the ideal would be to switch **dynamically** from one condition to the other **during the computation**.

Object falling at supersonic speed

Consider, for simplicity, a two dimensional case of an homogeneous ellipse in free fall. As the body accelerates, the pitching moments tend to increase the angle of attack until it stalls (A), and then the body starts to fall towards its other end and accelerating etc... (**"flutter"**). However, if the body has a large angular moment at (B) then it may happen that it rolls on itself, keeping always the same sense of rotation. This kind of falling mechanism is called **"tumbling"** and is characteristic of less slender and more massive objects.

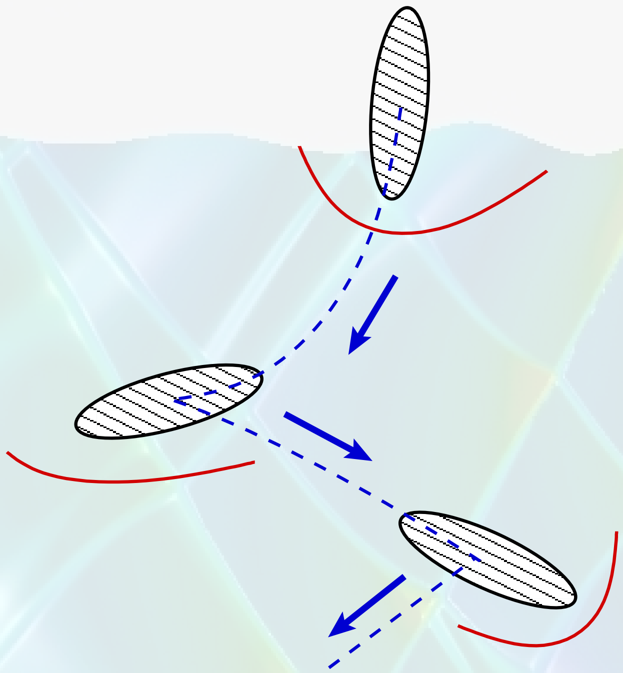


Object falling at supersonic speed (cont.)

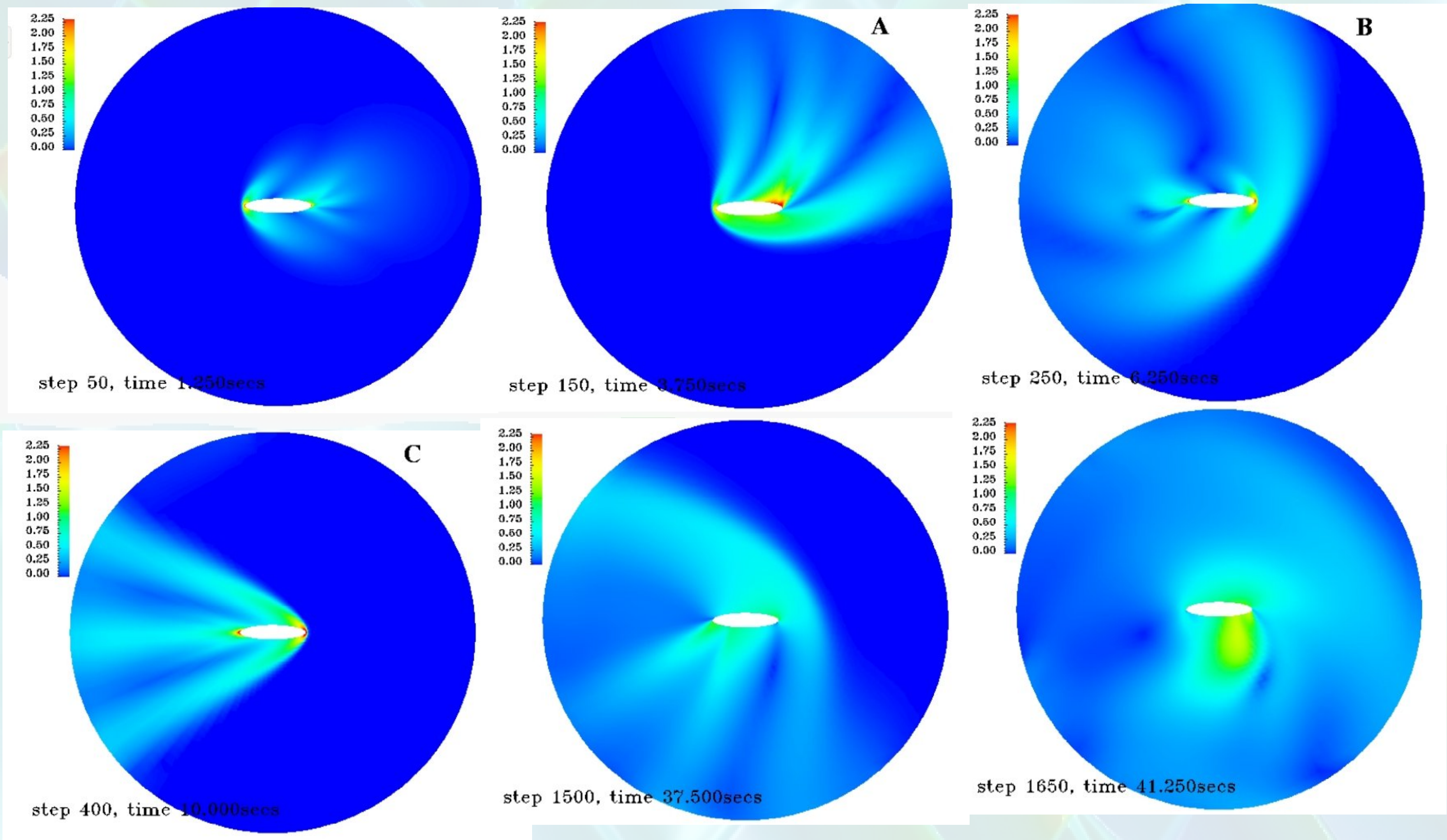
Under certain conditions in size and density relation to the surrounding atmosphere it reaches supersonic speeds. In particular as form drag grows like L^2 whereas weight grows like L^3, larger bodies tend to reach larger limit speeds and eventually reach supersonic regime. At supersonic speeds the principal source of drag is the shock wave, we use slip boundary condition at the body in order to simplify the problem.

We also do the computation in a non-inertial system following the body, so that non-inertial terms (Coriolis, centrifugal, etc...) are added. In this frame some portions of the boundary are alternatively in all the conditions (subsonic incoming, subsonic outgoing, supersonic incoming, supersonic outgoing).

Again, the ideal would be to switch dynamically from one condition to the other during the computation.



Object falling at supersonic speed (cont.)



Object falling at supersonic speed (cont.)

Whether RI or ULSAR based boundary conditions are used, if the number of incoming/outgoing characteristics vary in time this requires to change the profile of the system matrix during time evolution. In order to do this we add dynamic boundary conditions either via Lagrange multipliers or penalization. However this techniques add extra bad conditioning to the system of equations so that special iterative methods are needed.

Object falling at supersonic speed (cont.)

$$\mathbf{C}\,\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\,\frac{\partial \mathbf{U}}{\partial x} = 0.$$

Consider for simplicity a linear system of advective equations discretized with centered (i.e. no upwind) finite differences

$$\mathbf{C}\,\frac{\mathbf{U}_0^{n+1} - \mathbf{U}_0^{n}}{\Delta t} + \mathbf{A}\,\frac{\mathbf{U}_1^{n+1} - \mathbf{U}_0^{n+1}}{h} = 0;$$
$$\mathbf{C}\,\frac{\mathbf{U}_k^{n+1} - \mathbf{U}_k^{n}}{\Delta t} + \mathbf{A}\,\frac{\mathbf{U}_{k+1}^{n+1} - \mathbf{U}_{k-1}^{n+1}}{2h} = 0, \quad k \geq 1$$

$k \geq 0$ **node index,** $n \geq 0$ **time index,** $h =$ **mesh size,** $\mathbf{C}, \mathbf{A}$ $=$**enthalpy and advective Jacobians.**

Object falling at supersonic speed (cont.)

Using Lagrange multipliers for imposing the boundary conditions leds to the following equations

$$\boldsymbol{\Pi}^{+}\,(\mathbf{U}_0 - \mathbf{U}_{\mathrm{ref}}) + \boldsymbol{\Pi}^{-}\,\boldsymbol{\lambda} = 0,$$

$$\mathbf{C}\frac{\mathbf{U}_0^{n+1} - \mathbf{U}_0^{n}}{\Delta t} + \mathbf{A}\,\frac{\mathbf{U}_1^{n+1} - \mathbf{U}_0^{n+1}}{h} + \mathbf{C}\boldsymbol{\Pi}^{+}\,\boldsymbol{\lambda} = 0;$$

$$\mathbf{C}\frac{\mathbf{U}_k^{n+1} - \mathbf{U}_k^{n}}{\Delta t} + \mathbf{A}\,\frac{\mathbf{U}_{k+1}^{n+1} - \mathbf{U}_{k-1}^{n+1}}{2h} = 0, \quad k \geq 1.$$

$\boldsymbol{\Pi}^{\pm}$ **is the projection operator onto incoming/outgoing waves,** $\boldsymbol{\lambda}$ **are the Lagrange multipliers.**

Using penalization

Add a small regularization term and then eliminate the Lagrange multipliers.

$$-\epsilon\boldsymbol{\lambda} + \mathbf{\Pi}^+ (\mathbf{U}_0 - \mathbf{U}_{\mathrm{ref}}) + \mathbf{\Pi}^- \boldsymbol{\lambda} = 0,$$

$$\mathbf{C}\frac{\mathbf{U}_0^{n+1} - \mathbf{U}_0^n}{\Delta t} + \mathbf{A}\,\frac{\mathbf{U}_1^{n+1} - \mathbf{U}_0^{n+1}}{h} + \mathbf{C}\mathbf{\Pi}^+ \boldsymbol{\lambda} = 0;$$

Eliminating the Lagrange multipliers λ we arrive to a boundary equation

$$\mathbf{C}\frac{\mathbf{U}_0^{n+1} - \mathbf{U}_0^n}{\Delta t} + \mathbf{A}\,\frac{\mathbf{U}_1^{n+1} - \mathbf{U}_0^{n+1}}{h} + (1/\epsilon)\mathbf{C}\mathbf{\Pi}^+(\mathbf{U}_0 - \mathbf{U}_{\mathrm{ref}}) = 0.$$

Absorbing boundary conditions and ALE

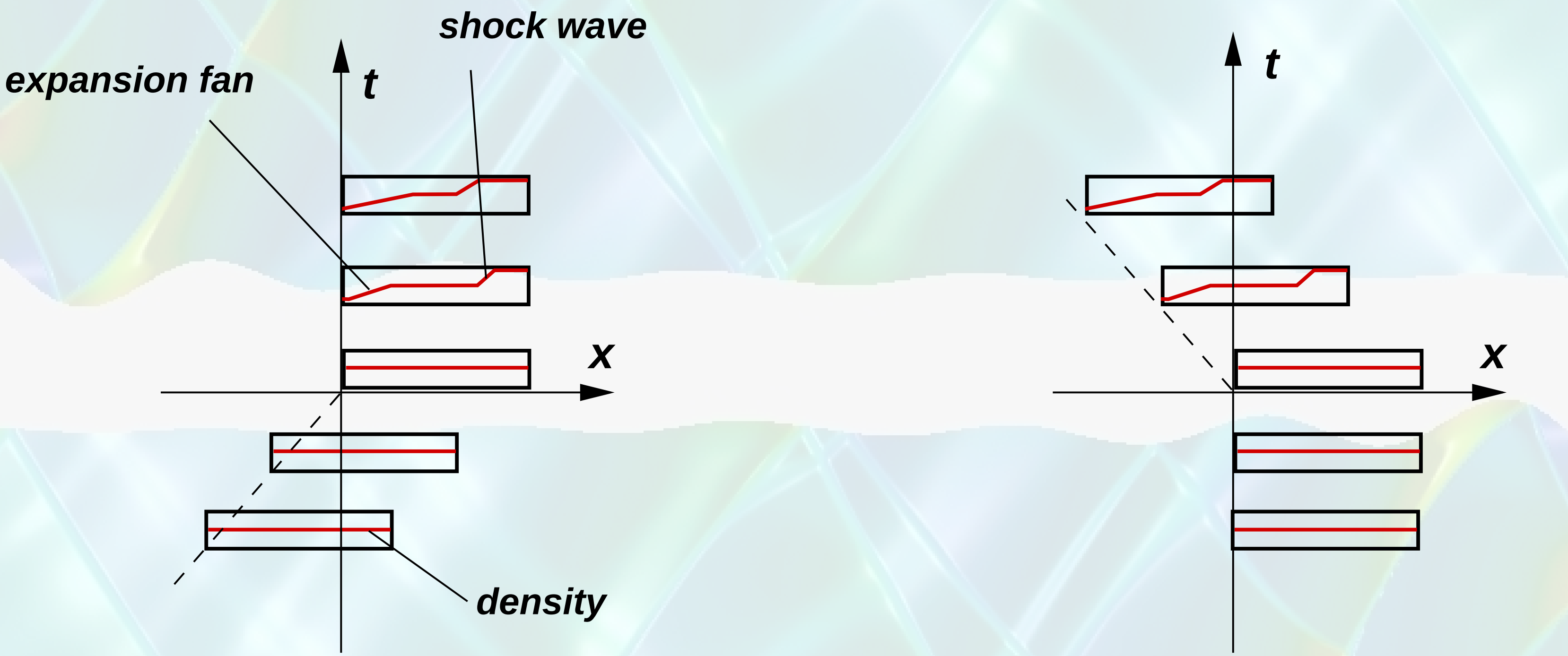
When using *Arbitrary Lagrangian-Eulerian* formulations:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathcal{F}_{c,j}(\mathbf{U})}{\partial x_j} - \mathbf{v}_{\mathrm{mesh}} \mathbf{U} = 0$$

$$A_{\mathrm{ALE},j} = \frac{\partial \mathcal{F}_{c,j}(\mathbf{U})}{\partial \mathbf{U}} - v_{\mathrm{mesh},j} \mathbf{I},$$ **ALE advective Jacobian**

Nbr. of incoming characteristics $= \mathrm{sum}(\mathrm{eig}(\mathbf{A} \cdot \hat{\mathbf{n}}) - \mathbf{v}_{\mathrm{mesh}} \cdot \hat{\mathbf{n}} < 0)$

ALE invariance test case. Sudden stop of gas container



container initially moving at constant speed u0 is suddenly stopped

container initially at rest is suddenly put in movement with constant negative speed −u0

ALE invariance. SUPG Stabilization term

$$\frac{\partial \mathbf{U}_c}{\partial t} + \frac{\partial \mathcal{F}_{c,x}}{\partial x} = \frac{\partial \mathcal{F}_{d,x}}{\partial x}; \qquad \textbf{(gov. eqs. in cons. form)}$$
$$C\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = \mathbf{K}\frac{\partial^2 \mathbf{U}}{\partial x^2}; \qquad \textbf{(gov. eqs. in quasi-linear form)} \tag{13}$$

Sufficient conditions for ALE invariance ($\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{v}_{\mathrm{mesh}}\mathbf{C}$)

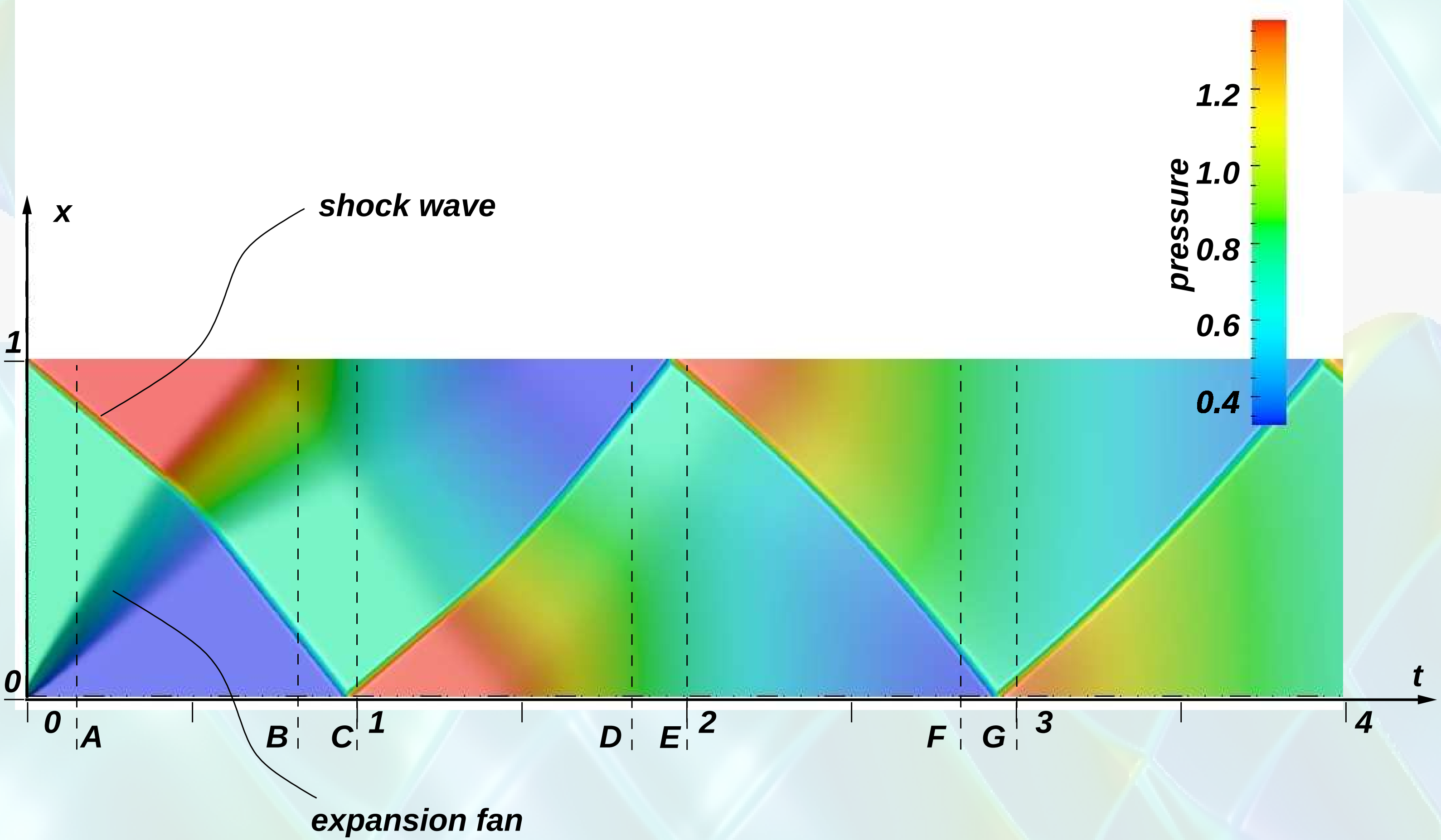
$$P = \nabla N \cdot \tilde{\mathbf{A}}\boldsymbol{\tau}\mathbf{C}^{-1}; \quad \textbf{(SUPG pert. function)}$$
$$\boldsymbol{\tau} \ \textbf{transform as } \mathbf{U}\times\mathbf{U}, \quad \left(\textbf{i.e. } \boldsymbol{\tau}' = \frac{\partial \mathbf{U}'}{\partial \mathbf{U}}\boldsymbol{\tau}\frac{\partial \mathbf{U}}{\partial \mathbf{U}'}\right). \tag{14}$$

This last is verified if $\boldsymbol{\tau}$ is $f(\mathbf{C}^{-1}\tilde{\mathbf{A}})$, for instance (inviscid case):

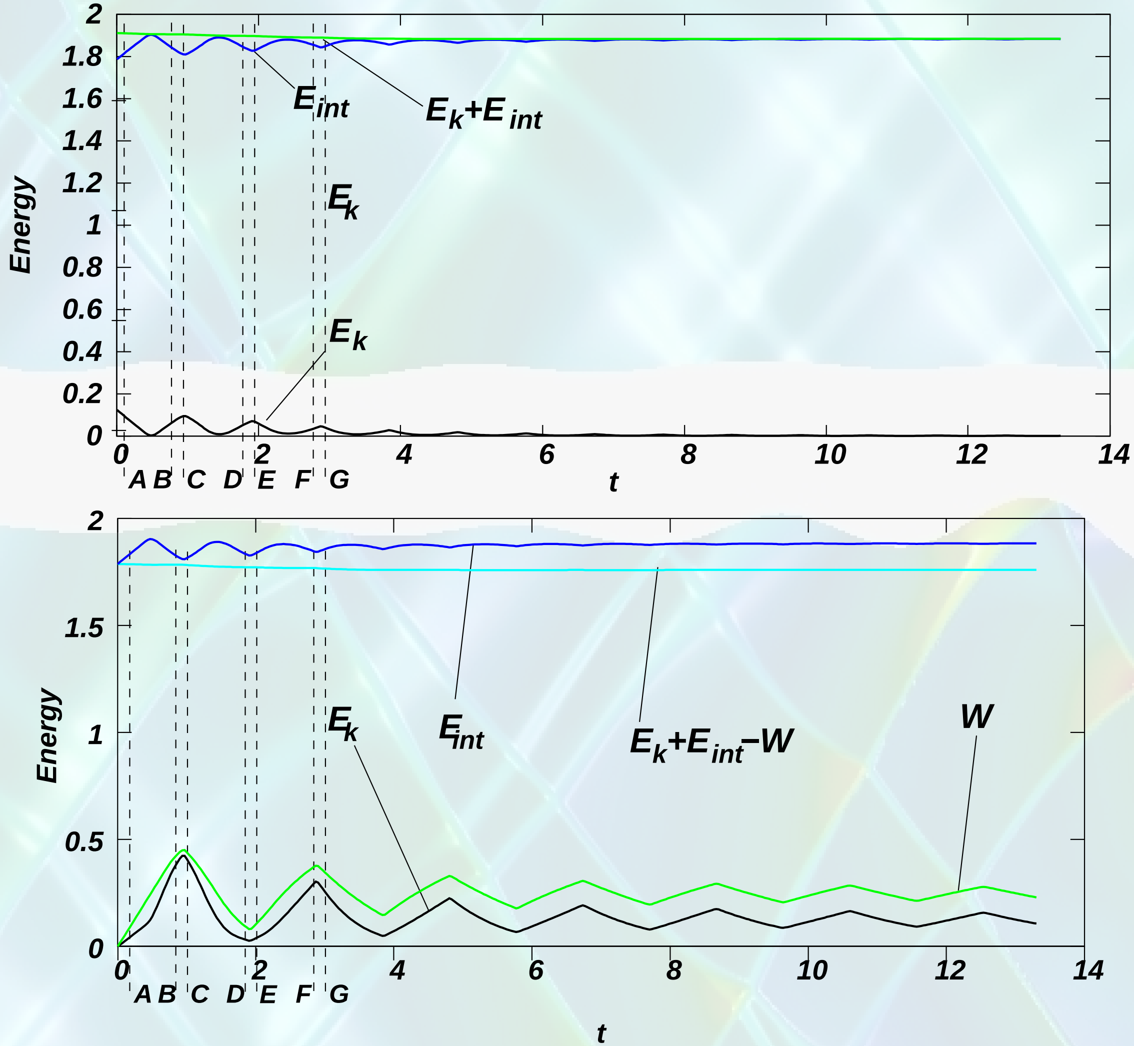
$$\boldsymbol{\tau} = \frac{h}{\max|\lambda_j|}\mathbf{I}, \ \ \lambda_j = \mathrm{eig}(\mathbf{C}^{-1}\tilde{\mathbf{A}}), \qquad \textbf{(max. eigenv.)}$$
$$\boldsymbol{\tau} = h|\mathbf{C}^{-1}\tilde{\mathbf{A}}|^{-1}. \qquad \textbf{(} | \cdot | \textbf{ in matrix sense)} \tag{15}$$

ALE invariance. Sudden stop of a gas container

$\gamma = 1.4$, $u_0/c_0 = 0.5$. **Results in both reference systems are equivalent to machine precision.**



ALE invariance. Sudden stop of a gas container (cont.)



(Up) Energy balance in reference system fixed w.r.t container)
(Down) Energy balance in reference system fixed w.r.t initial gas at rest)

Open channel flow.

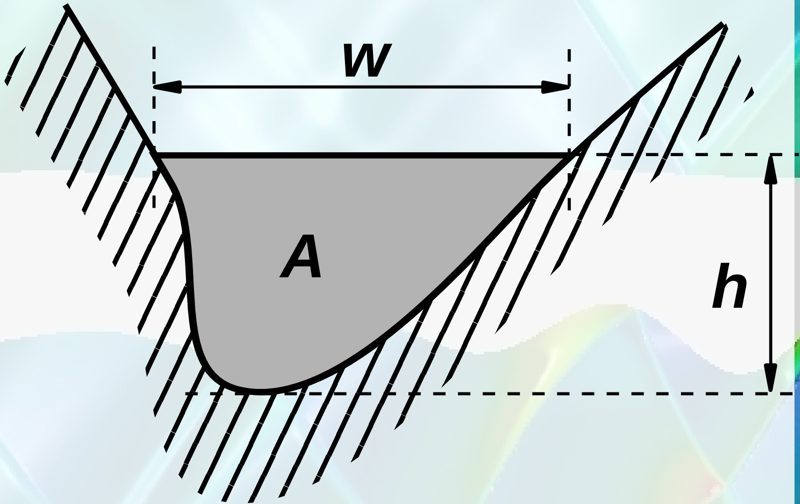
Flow in a channel can be cast in advective form as follows

$$\mathbf{U}_p = [h, u]^T,$$

$$\mathbf{U} = \mathbf{U}_c = [A, Q]^T,$$

$$\mathcal{H}(\mathbf{U}) = \mathbf{U}_c,$$

$$\mathcal{F} = \begin{bmatrix} Q \\ Q^2/A + F \end{bmatrix}.$$



where h and u are water depth and velocity (as in the shallow water equations).

$A(h)$ is the section of the channel occupied by water for a given water height h. It then defines the geometry of the channel. $Q = Au$ is the water flow rate.

Open channel flow. (cont.)

For instance

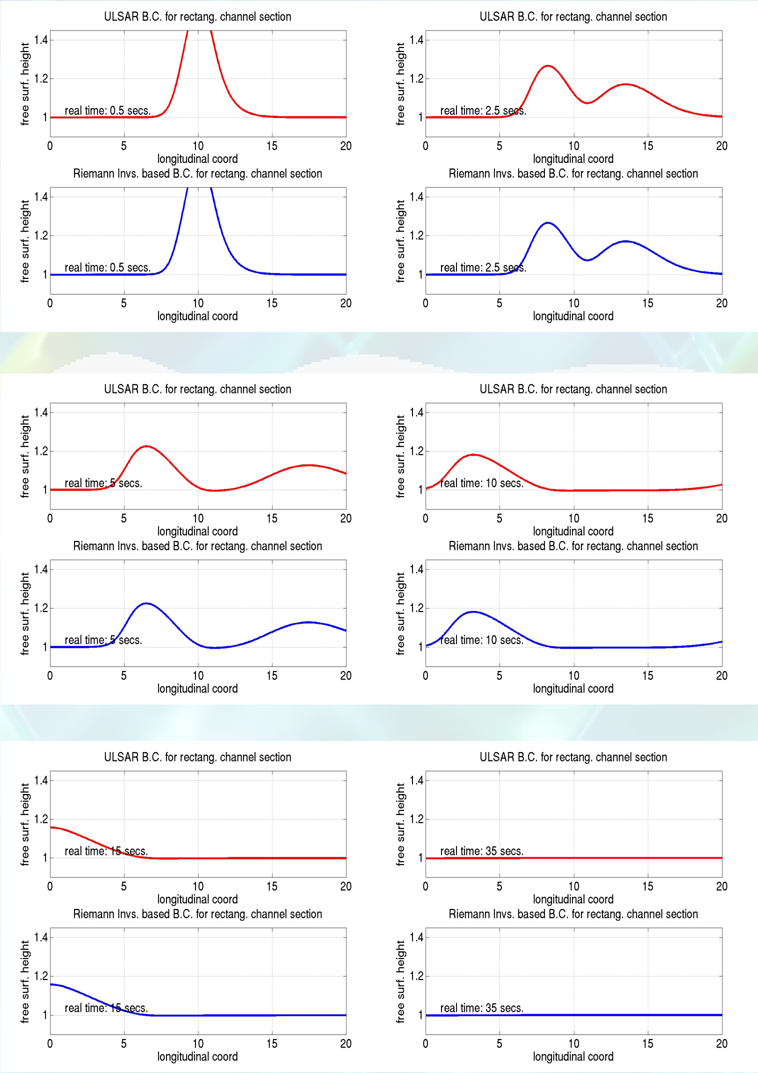
- **Rectangular channels:** $A(h) = wh,\ w$**=width.**
- **Triangular channels:** $A(h) = 2h^2 \tan\theta/2$**; with** θ**=angle opening.**
- **Circular channel:**

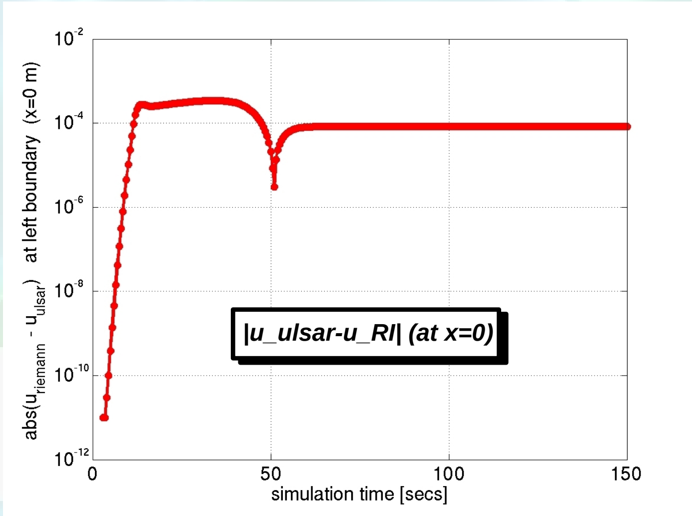
$$
\begin{aligned}
A(h) &= \int_{h'=0}^{h} \sqrt{2Rh - h^2}\,\mathrm{d}h' \\
&= \theta R^2 - w(h)(R-h)/2
\end{aligned}
$$

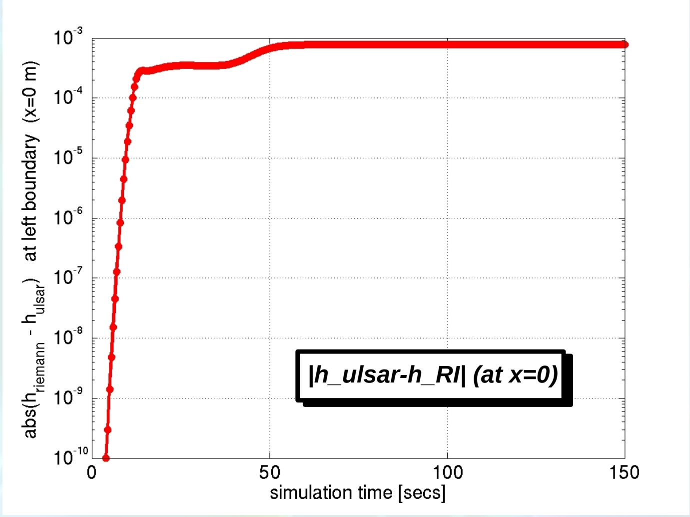
$w(h) = 2\sqrt{2Rh - h^2}$**,**
$\theta = \mathrm{atan}[w/(2(R-h))]$ **is angular aperture,**
$F(h) = \int_{h'=0}^{h} A(h')\,\mathrm{d}h'$

- **Double rectangular channel.**

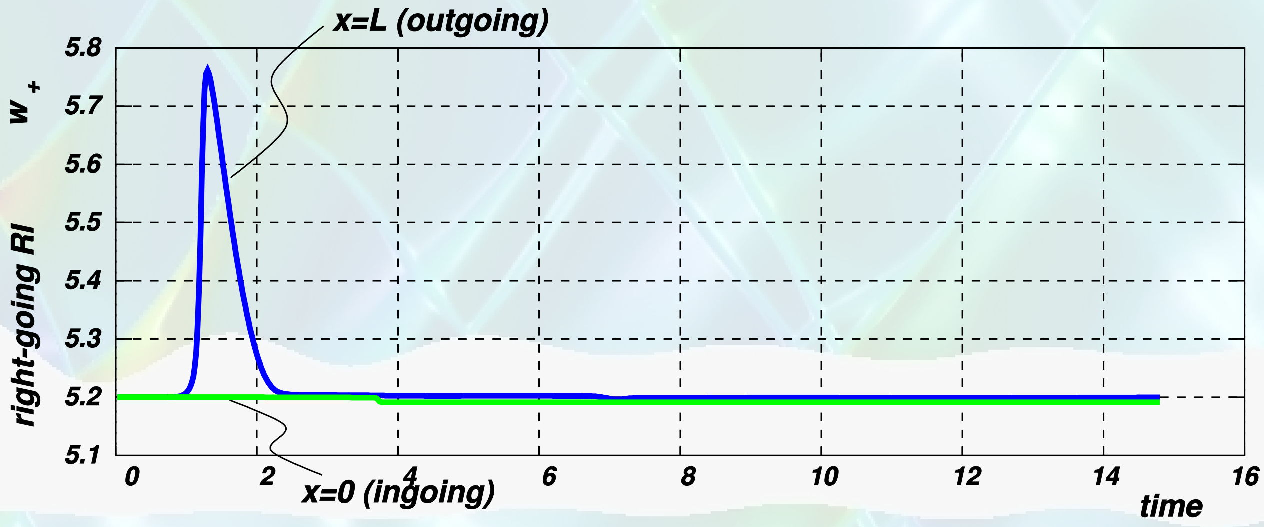
Comparison with RI for rectangular channel

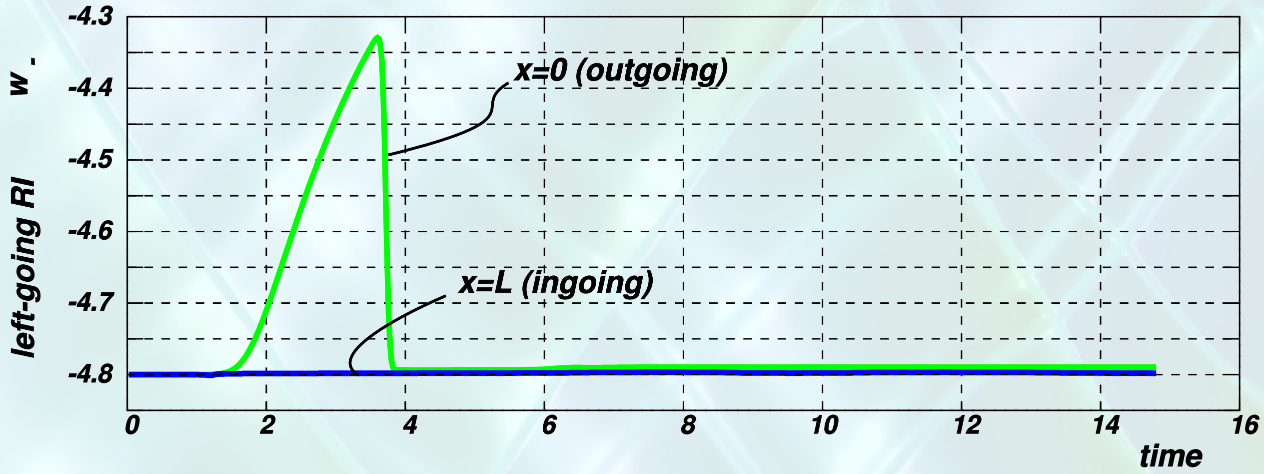






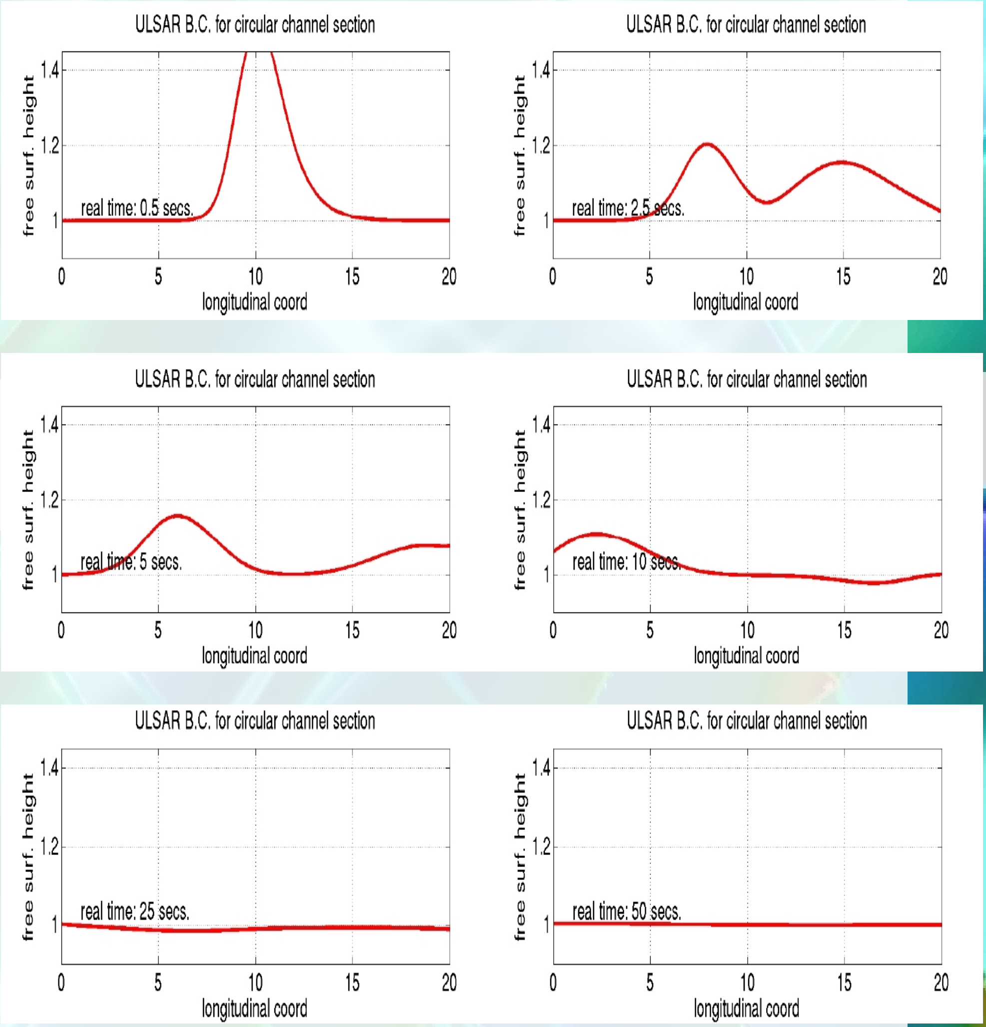
Comparison with RI for rectangular channel (cont.)





Comparison with RI for rectangular channel (cont.)

- We restrict here to the case of ***constant channel section and depth***.
- For rectangular channels the equations are reduced to those for 1D shallow water equations.
- Channel flow is very interesting since it is in fact a ***family of different 1D hyperbolic systems*** depending on the area function $A(h)$.
- ***Riemann invariants are only known for rectangular and triangular channel shapes.***



Stratified shallow water flows

Another physical model where *Riemann invariants* have not a mathematical closed form is the flow of a *multi-layer fluid in channels*. This kind of physical model exists for instance when flow takes place on a mountainous terrain over plain areas or dense distribution of torrents combined with heavy rainfall. Each layer may have different density, velocity in a two-dimensional domain.

$$\frac{\partial \mathbf{U}_c}{\partial t} + \frac{\partial \mathcal{F}_{c,j}(\mathbf{U}_c)}{\partial x_j} + \mathbf{B}_j(\mathbf{U}_c)\frac{\partial \mathbf{U}_c}{\partial x_j} = \mathbf{G}(\mathbf{U}_c); \qquad j = 1, 2. \tag{16}$$

$\mathbf{U}_c = [h_1, h_2, h_1u_1, h_1v_1, h_2u_2, h_2v_2]^T$ is the vector of conservation variables and $\mathbf{B}_j(\mathbf{U}_c)$ is the Jacobian matrix of the non-conservative products in the j direction. $h_1(\mathbf{x}, t)$ and $h_2(\mathbf{x}, t)$ are the thickness of each layer while the height of the bottom is $h_0(\mathbf{x}, t)$. $[u, v]_i$ is the velocity vector of the layer i.

Startified shallow water eqs (two strata)

$$
\begin{aligned}
\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}(h_1 u_1) + \frac{\partial}{\partial y}(h_1 v_1) &= 0, \\
\frac{\partial h_2}{\partial t} + \frac{\partial}{\partial x}(h_2 u_2) + \frac{\partial}{\partial y}(h_2 v_2) &= 0, \\
\frac{\partial}{\partial t}(h_1 u_1) + \frac{\partial}{\partial x}\left(h_1 (u_1)^2 + \frac{1}{2} g h_1^2\right) + \frac{\partial}{\partial y}(h_1 u_1 v_1) + g h_1 \frac{\partial}{\partial x}\left(h_0 + \frac{\rho_2}{\rho_1} h_2\right) &= 0, \\
\frac{\partial}{\partial t}(h_1 v_1) + \frac{\partial}{\partial x}(h_1 u_1 v_1) + \frac{\partial}{\partial y}\left(h_1 (v_1)^2 + \frac{1}{2} g h_1^2\right) + g h_1 \frac{\partial}{\partial y}\left(h_0 + \frac{\rho_2}{\rho_1} h_2\right) &= 0, \\
\frac{\partial}{\partial t}(h_2 u_2) + \frac{\partial}{\partial x}\left(h_2 (u_2)^2 + \frac{1}{2} g h_2^2\right) + \frac{\partial}{\partial y}(h_2 u_2 v_2) + g h_2 \frac{\partial}{\partial x}(h_0 + h_1) &= 0, \\
\frac{\partial}{\partial t}(h_2 v_2) + \frac{\partial}{\partial x}(h_2 u_2 v_2) + \frac{\partial}{\partial y}\left(h_2 (v_2)^2 + \frac{1}{2} g h_2^2\right) + g h_2 \frac{\partial}{\partial y}(h_0 + h_1) &= 0,
\end{aligned}
$$

Stratified shallow water flows

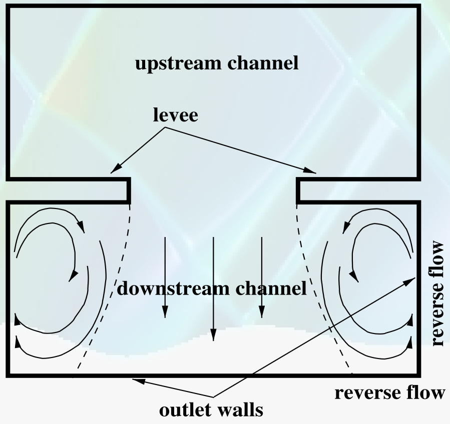
- $\rho_{1,2}$ are constant layer densities.
- Vertical velocity is averaged.
- Hydrostatic pressure assumption in each layer.

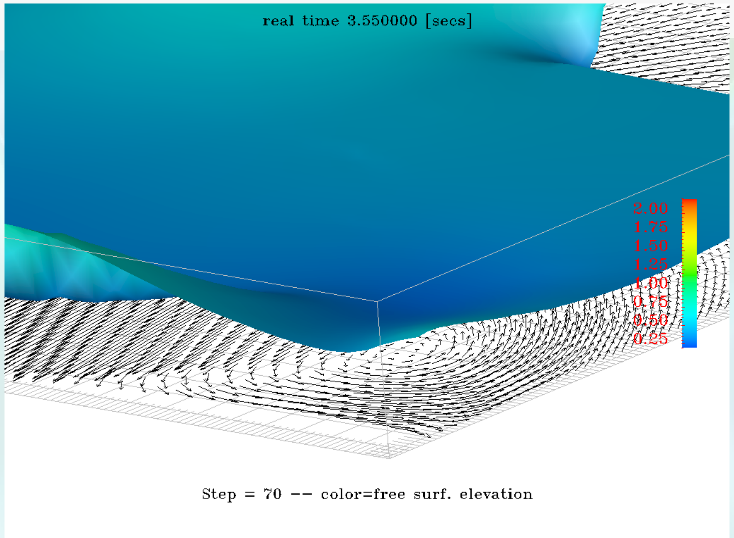
In the general case of n layers and 2D model, the system of equations has $3n$ waves that propagate inside the domain.

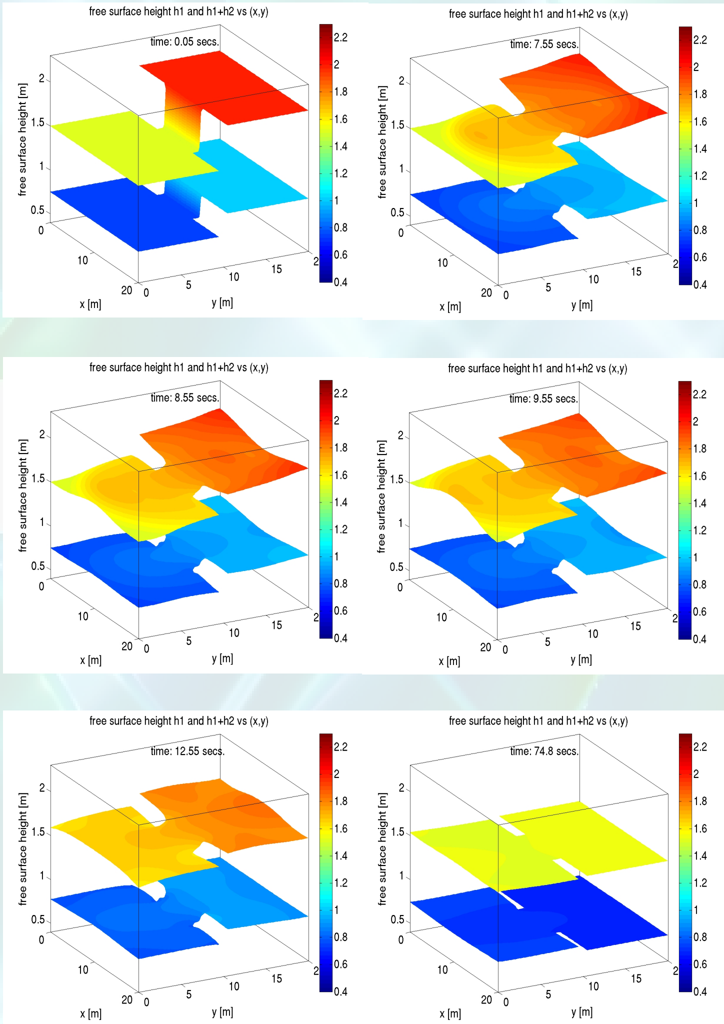
It could happen that, depending on the densities ratio ρ_2/ρ_1, and the velocities the system becomes ***non-hyperbolic***. In fact if the velocities of the fluid are the same, the flow is unstable for $\rho_2 > \rho_1$.

Riemann-Invariants for stratified flow are not known.

Stratified shallow water flows (cont.)







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