UNL

Universidad Nacional del Litoral

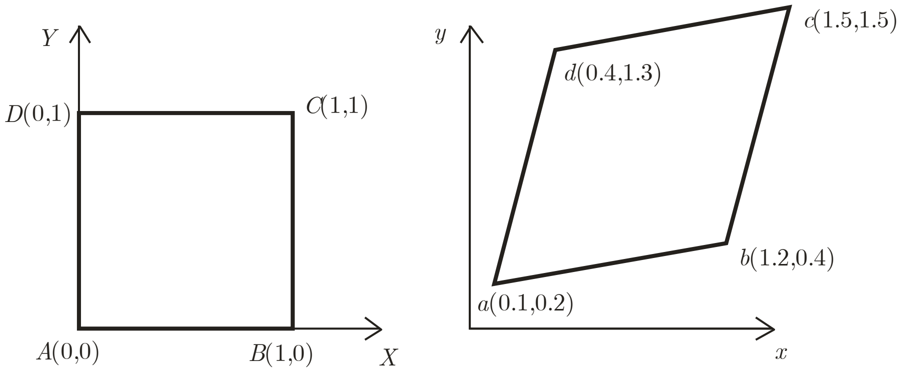
Facultad de Ingeniería y Ciencias Hídricas

Examen Parcial 24/06/2013

1. Sea el cuadrado ABCD que representa la configuración inicial de un cuerpo. El cuerpo está sujeto a un cambio de configuración lineal (deformación) de la forma:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} a_0 + a_1 X + a_2 Y \\ b_0 + b_1 X + b_2 Y \end{Bmatrix}$$

Tras la deformación, el punto A ocupa la posición a, B ocupa la posición b, C ocupa la posición c y D ocupa la posición d, como se indica en la figura.



   a. Calcule el campo de desplazamiento $\mathbf{u}(X,Y)$.

   b. Calcule el campo tensorial de deformación de Green-Lagrange $\mathbf{E}(X,Y)$.

   c. Calcule el campo tensorial de deformaciones infinitesimales $\boldsymbol{\varepsilon}(X,Y)$.

   d. En el caso (c), calcule las deformaciones principales infinitesimales y direcciones principales para el punto de coordenadas $X = 0.5;\quad Y = 0.5$.

2. La forma más general de un tensor isótropo de 4° rango es la siguiente:

$$C_{ijkl} = a_1\delta_{ij}\delta_{kl} + a_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + a_3(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

donde a_1, a_2, a_3 son constantes reales.

   a. ¿Cuál es la forma que toma C_{ijkl} , para el caso del tensor de elasticidad de un sólido elástico lineal e isótropo?, ¿por qué?

   b. Dé un significado a las constantes a_i que aparecen en la expresión del tensor de elasticidad de un sólido elástico lineal e isótropo.

3. Sea V el volumen encerrado por una superficie S de normal saliente unitaria n, y sean $\phi = \phi(x_1, x_2, x_3)$ y $\psi = \psi(x_1, x_2, x_3)$ funciones escalares de las coordenadas x_i. Usando el teorema de Green, demostrar:

$$\int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\boldsymbol{n}\ dS = \int_V (\phi\Delta\psi - \psi\Delta\phi)\ dV$$

donde $\Delta\psi = \psi_{,ii}$ es el laplaciano de la función ψ.

1)
a) $x = \alpha_0 + \alpha_1 X + \alpha_2 Y$

$y = \beta_0 + \beta_1 X + \beta_2 Y$

$x(0,0) = \alpha_0 = 0.1 \Rightarrow \alpha_0 = 0.1$

$y(0,0) = \beta_0 = 0.2 \Rightarrow \beta_0 = 0.2$

$x(1,0) = 0.1 + \alpha_1 = 1.2 \Rightarrow \alpha_1 = 1.1$

$x(0,1) = 0.1 + \alpha_2 = 0.4 \Rightarrow \alpha_2 = 0.3$

$y(1,0) = 0.2 + \beta_1 = 0.4 \Rightarrow \beta_1 = 0.2$

$y(0,1) = 0.2 + \beta_2 = 1.3 \Rightarrow \beta_2 = 1.1$

Verif:

$x(1,1) = 0.1 + 1.1 + 0.3 = 1.5$ ✓

$y(1,1) = 0.2 + 0.2 + 1.1 = 1.5$ ✓

$$\underline{u} = \underline{x} - \underline{X} = \begin{pmatrix} 0.1 + 1.1X + 0.3Y - X \\ 0.2 + 0.2X + 1.1Y - Y \end{pmatrix} = \begin{pmatrix} 0.1 + 0.1X + 0.3Y \\ 0.2 + 0.2X + 0.1Y \end{pmatrix}$$

$$\underline{u} = \begin{pmatrix} 1 + X + 3Y \\ 2 + 2X + Y \end{pmatrix} \times 1/10$$

b)

$\underline{\underline{E}}(X,Y) = ?$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_\alpha}{\partial X_i}\frac{\partial u_\alpha}{\partial X_j}\right)$$

$$\frac{\partial u_1}{\partial X_1} = 1/10 \quad \frac{\partial u_1}{\partial X_2} = \frac{3}{10} \quad \frac{\partial u_2}{\partial X_1} = \frac{2}{10} \quad \frac{\partial u_2}{\partial X_2} = \frac{1}{10}$$

②

$$E_{11} = \frac{1}{2}\left(\frac{1}{10} + \frac{1}{10} + \frac{1}{10}\times\frac{1}{10} + \frac{2}{10}\times\frac{2}{10}\right) =$$

$$= \frac{1}{2} \quad \frac{10+10+1+4}{100} = \frac{25}{200} = \frac{1}{8}$$

$$E_{22} = \frac{1}{2}\left(\frac{1}{10} + \frac{1}{10} + \frac{3}{10}\times\frac{3}{10} + \frac{1}{10}\times\frac{1}{10}\right) =$$

$$= \frac{1}{2}\left(\frac{10+10+9+1}{100}\right) = \frac{30}{200} = \frac{3}{20}$$

$$E_{12} = E_{21} = \frac{1}{2}\left(\frac{3}{10} + \frac{2}{10} + \frac{1}{10}\,\frac{3}{10} + \frac{2}{10}\,\frac{1}{10}\right) =$$

$$= \frac{1}{2}\left(\frac{30+20+3+2}{100}\right) = \frac{55}{200}$$

$$\underline{\underline{E}} = \begin{bmatrix} 1/8 & 55/200 \\ 55/200 & 3/20 \end{bmatrix} = \begin{bmatrix} 0.125 & 0.275 \\ 0.275 & 0.15 \end{bmatrix}$$

c) $\underline{\underline{e}}(X,Y) = ?$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

$$e_{11} = \frac{1}{10} \qquad e_{12} = e_{21} = \frac{5}{20} = \frac{1}{4}$$

$$e_{22} = \frac{1}{10}$$

$$\underline{\underline{e}} = \begin{bmatrix} 0.1 & 0.25 \\ 0.25 & 0.1 \end{bmatrix}$$

INDEPENDIENTE DEL PUNTO DE EVALUACION

d)

$$\det \begin{pmatrix} 0.1-\lambda & 0.25 \\ 0.25 & 0.1-\lambda \end{pmatrix} = 0$$

$$0.1^2 - 2 \times \lambda \times 0.1 + \lambda^2 - 0.25^2 = 0$$

$$\frac{1}{100} - \frac{2}{10}\lambda + \lambda^2 - \frac{1}{16} = 0$$

$$\lambda^2 - \frac{2}{10}\lambda + \frac{16-100}{1600} = 0$$

$$\lambda^2 - \frac{2}{10}\lambda - \frac{84}{1600} = 0$$

$$1600\lambda^2 - 320\lambda - 84 = 0$$

$$\lambda_{1,2} = \frac{320 \pm \sqrt{320^2 + 4 \times 84 \times 1600}}{3200} =$$

$$= \frac{1}{10} \pm \frac{\sqrt{102400 + 537600}}{3200} = \frac{1}{10} \pm \frac{1}{4}$$

$$\lambda_1 = 0.35 \qquad \lambda_2 = -0.15$$

$$\begin{pmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1^1 = x_2^1$$

$$\therefore \left(\lambda_1, \underline{x}^1\right) = \left(0.35, \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}\right)$$

$$\begin{pmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1^2 = -x_2^2$$

$\left(\sqrt{2}/2\right)$

2)

a) $C_{ijkl} = a_1 \delta_{ij}\delta_{kl} + a_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + a_3(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$ (1)

Para un material elástico:

$$\sigma_{ij} = C_{ijkl}\, e_{kl}$$

donde $\sigma_{ij} = \sigma_{ji}$ y $e_{kl} = e_{lk}$

Dada la simetría de $\underline{\underline{\sigma}}$:

$$\sigma_{ij} = C_{ijkl}\, e_{kl} = \sigma_{ji} = C_{jikl}\, e_{kl} \quad \forall e_{kl}$$

Luego, debe ser

$$C_{ijkl} = C_{jikl}$$

En la expresión (1):

(A) $C_{ijkl} = a_1 \delta_{ij}\delta_{kl} + a_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + a_3(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$

(B) $C_{jikl} = a_1 \delta_{ji}\delta_{kl} + a_2(\delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik}) + a_3(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik})$

Hago A - B :

$$0 = 2 a_3 (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

$$\therefore \ a_3 = 0$$

$$\therefore \ C_{ijkl} = a_1 \delta_{ij}\delta_{kl} + a_2 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

b) Para un sólido lineal isótropo:

$$\sigma_{ij} = C_{ijkl}\, e_{kl} = \lambda\, e_{\alpha\alpha}\, \delta_{ij} + 2\mu\, e_{ij}$$

Pero

$$C_{ijkl}\, e_{kl} = a_1 \delta_{ij}\delta_{kl}\, e_{kl} + a_2 (\delta_{ik}\delta_{jl})\, e_{kl} + $$
$$+ a_2 \delta_{il}\delta_{jk}\, e_{kl} =$$
$$= a_1 e_{kk}\delta_{ij} + a_2 e_{ij} + a_2 e_{ji} =$$
$$= a_1 e_{\alpha\alpha}\delta_{ij} + 2 a_2 e_{ij}$$

Porque $e_{ij} = e_{ji}$

Luego

$$a_1 = \lambda$$
$$a_2 = \mu$$

(Ctes de Lamé)

$\int_S (\ldots) = \int_V \ldots$

Ⓐ $$\int_S \phi \frac{\partial \psi}{\partial x_i} n_i \, dS \underset{\text{por Teorema Gauss}}{=} \int_V \frac{\partial}{\partial x_i}\left(\phi \frac{\partial \psi}{\partial x_i}\right) dV =$$

$$= \int_V \left(\frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_i} + \phi \frac{\partial^2 \psi}{\partial x_i \partial x_i} \right) dV$$

Ⓑ $$\int \psi \frac{\partial \phi}{\partial x_i} n_i \, dS = \int_V \left(\frac{\partial \psi}{\partial x_i} \frac{\partial \phi}{\partial x_i} + \psi \frac{\partial^2 \psi}{\partial x_i \partial x_i} \right) dV$$

Ⓐ-Ⓑ) $$\int_S \left(\phi \frac{\partial \psi}{\partial x_i} - \psi \frac{\partial \phi}{\partial x_i} \right) n_i \, dS =$$

$$= \int_V \left(\cancel{\frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_i}} + \phi \frac{\partial^2 \psi}{\partial x_i \partial x_i} - \cancel{\frac{\partial \psi}{\partial x_i} \frac{\partial \phi}{\partial x_i}} - \psi \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right) dV$$

$$\therefore \int_S \left(\phi \frac{\partial \psi}{\partial x_i} - \psi \frac{\partial \phi}{\partial x_i} \right) n_i \, dS = \int_V \left(\phi \frac{\partial^2 \psi}{\partial x_i \partial x_i} - \psi \frac{\partial^2 \phi}{\partial x_i \partial x_i} \right) dV$$