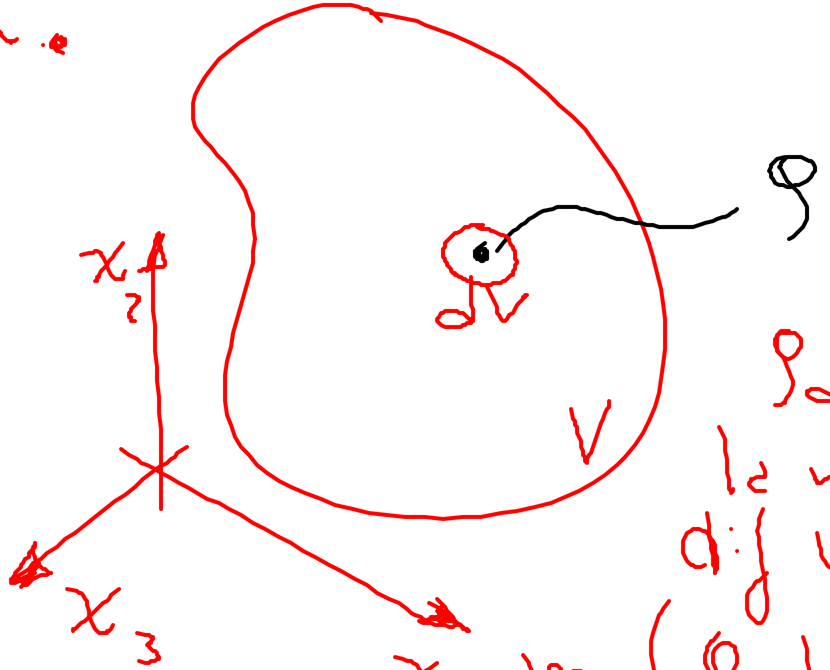
TEMA 5: Mov cuerpo rígido

CR: sistema de partículas / las distancias entre partículas no varían.

Es una aproximación.

Los cuerpos sólidos muchas veces verifican esta condición.

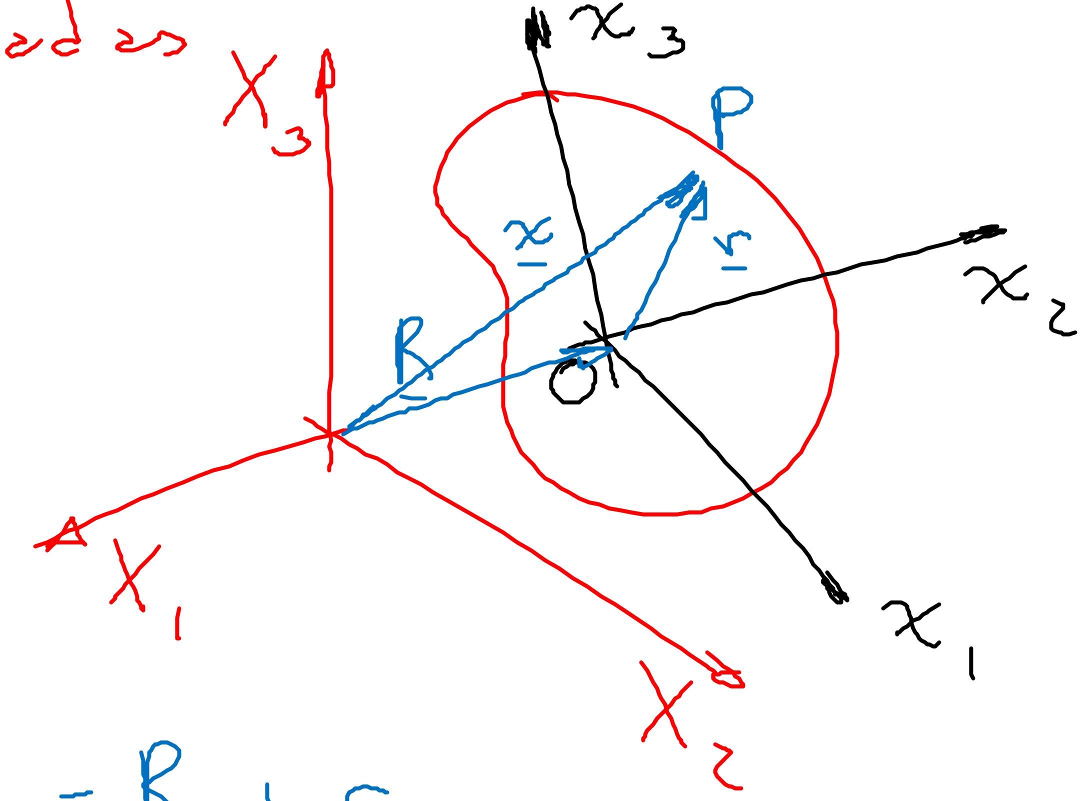


$\rho\, dV$ será la masa del dif volumen

$m = \int \rho\, dV$

Cuerpos sólidos
- Rígidos
- Flexibles
  - Elasticidad
  - Plasticidad
  - VPlast

P/ descripción del movimiento, usamos dos sistemas de coordenadas



$X_1 X_2 X_3$: sistema "fijo" o inercial "coord espaciales"

$O x_1 x_2 x_3$: sistema fijo en el cuerpo "coord materiales"

O : fijado en el centro de masa del cuerpo.

$$\underline{x} = \underline{R} + \underline{r}$$

P/ determinar la posición de P, necesitamos conocer coord del CM ($\underline{R}$) y la orientación del ($O x_1 x_2 x_3$).

La orientación del cuerpo la podemos determinar, por ejemplo, conociendo 3 ángulos de Euler (u otro sist)

luego, el CR tiene 6 GDL.

Consideremos un movimiento infinitesimal arbitrario del CR.

Lo pensamos compuesto P/ 2 partes:

i) traslación infinitesimal c/orientación fija

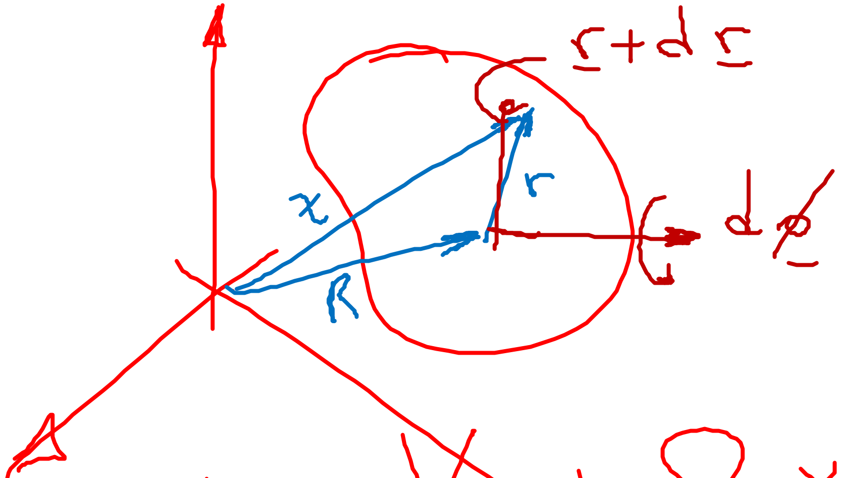
ii) rotación infinitesimal entorno al centro de masa.

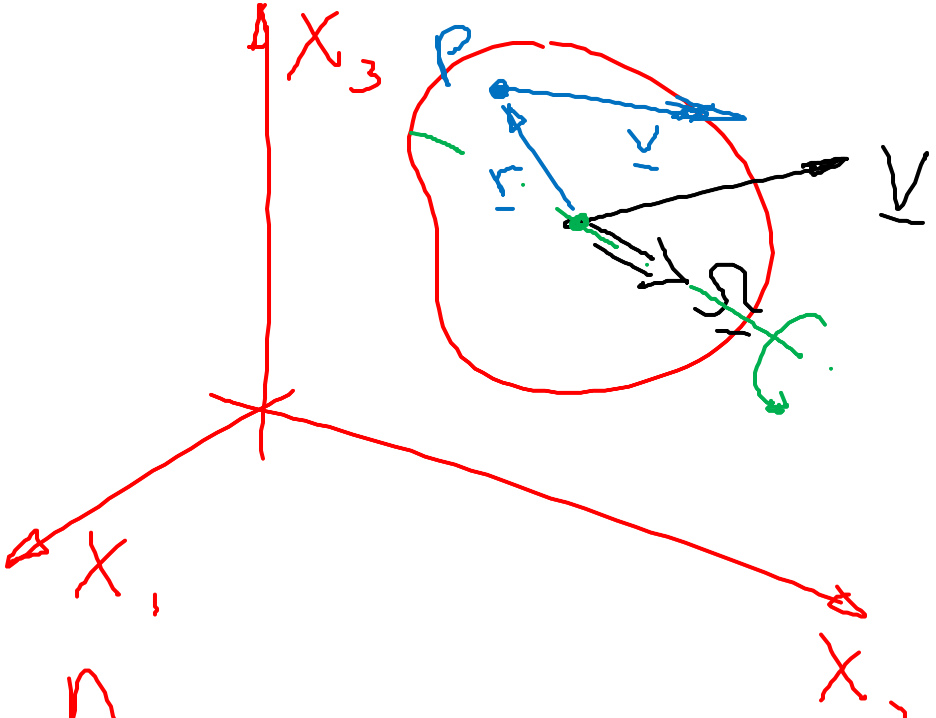
$$\underline{x} = \underline{R} + \underline{r}$$

$$d\underline{x} = d\underline{R} + d\underline{\phi} \times \underline{r}$$

Div x dt:

$$\frac{d\underline{x}}{dt} = \frac{d\underline{R}}{dt} + \frac{d\underline{\phi}}{dt} \times \underline{r} \Rightarrow \underline{v} = \underline{V} + \underline{\Omega} \times \underline{r}$$





$$\underline{v} = \underline{V} + \underline{\Omega} \times \underline{r}$$

$\underline{V}$: vel de traslación del CR

$\underline{\Omega}$: vel angular de rotación del CR

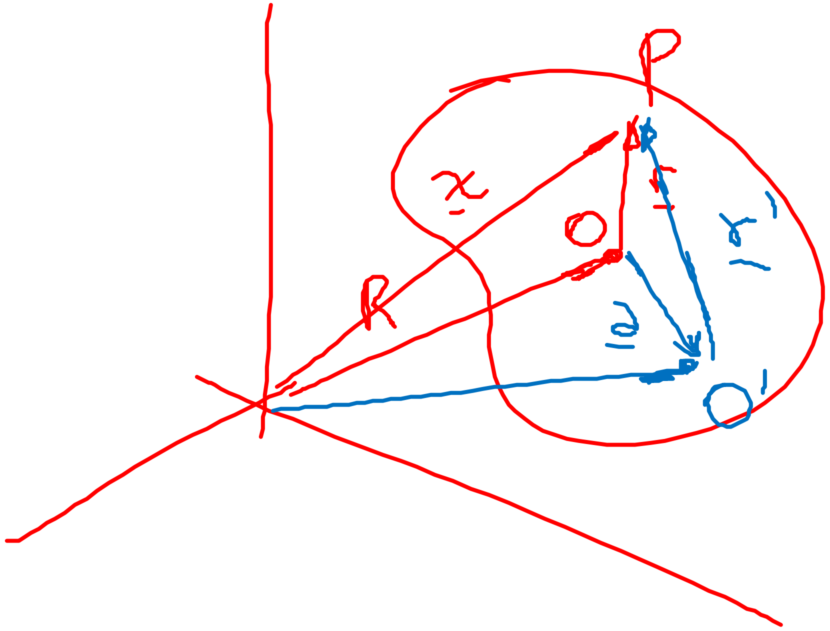
Asumimos origen de coord del sist material está en un punto O' a una dist $\underline{a}$ del punto O (Centro de masa).

Sea $\underline{V}'$ la vel de O'

$\underline{\Omega}'$ la vel angular del nuevo sist

$\underline{r}'$ la posición de P en el nuevo sist

$$\underline{r} = \underline{r}' + \underline{a}$$



$$\underline{r} = \underline{a} + \underline{r}'$$

En la anterior: ⊛

$$\underline{v} = \underline{V} + \underline{\Omega} \times \underline{r} =$$

$$= \underbrace{\underline{V} + \underline{\Omega} \times \underline{a}}_{\underline{V}'} + \underline{\Omega} \times \underline{r}'$$

Entonces:

$$\underline{v} = \underline{V}' + \underline{\Omega} \times \underline{r}'$$

y por comparación con ⊛

Concluyo que $\underline{\Omega}' = \underline{\Omega}$

Todo punto del CR está animado de la misma vel de rotación.

Luego, hablo de "velocidad angular del cuerpo rígido".

De la expresión:

$$\underline{V}' = \underline{V} + \underline{\Omega} \times \underline{a}$$

vemos q/si $\underline{V} \perp \underline{\Omega}$, luego $\underline{V}'$ y $\underline{\Omega}'$ son perpendiculares ∀ cualquier elección de O'.

$$\left(\underline{V}' = \underline{V} + (\underline{V}' - \underline{V}) \quad y \quad (\underline{V}' - \underline{V}) = \underline{\Omega} \times \underline{a} \perp \underline{\Omega} \right)$$

Luego, es posible encontrar $\underline{a}$ / $\underline{V}' = \underline{0}$

$$\underline{V} + \underline{\Omega} \times \underline{a} = 0$$

$$\underline{V} + \underline{\Omega} \times \underline{a} = \underline{0} \Rightarrow \tilde{\underline{\underline{\Omega}}}\,\underline{a} = -\underline{V}$$

$$\tilde{\underline{\underline{\Omega}}} = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix}$$

$\tilde{\underline{\underline{\Omega}}}$: matriz antisimétrica ✓

$$\tilde{\underline{\underline{\Omega}}}\,\underline{a} = \underline{\Omega} \times \underline{a} \quad \forall\, \underline{a}$$

$$\underline{V} \cdot \underline{\Omega} = 0 \Rightarrow \underline{V}' \cdot \underline{\Omega}' = 0$$

Si $\underline{a}$ es solución de (*), luego $(\underline{a} + \underline{\Omega}\alpha)$ también lo es:

$$\tilde{\underline{\underline{\Omega}}}\,(\underbrace{\underline{a} + \alpha\underline{\Omega}}_{\underline{a}^*}) = \tilde{\underline{\underline{\Omega}}}\,\underline{a} + \alpha \underbrace{\tilde{\underline{\underline{\Omega}}}\,\cancel{\times}\underline{\Omega}}_{0} = -\underline{V}$$

Sea α / $\underline{a}^* \cdot \underline{\Omega} = 0$

$$\begin{bmatrix} \underset{=}{\tilde{\Omega}} \\ \underline{\Omega}^T \end{bmatrix} \underline{a} = \begin{pmatrix} -\underline{V} \\ 0 \end{pmatrix}$$

$$\tilde{\Omega} \qquad \Omega^T$$

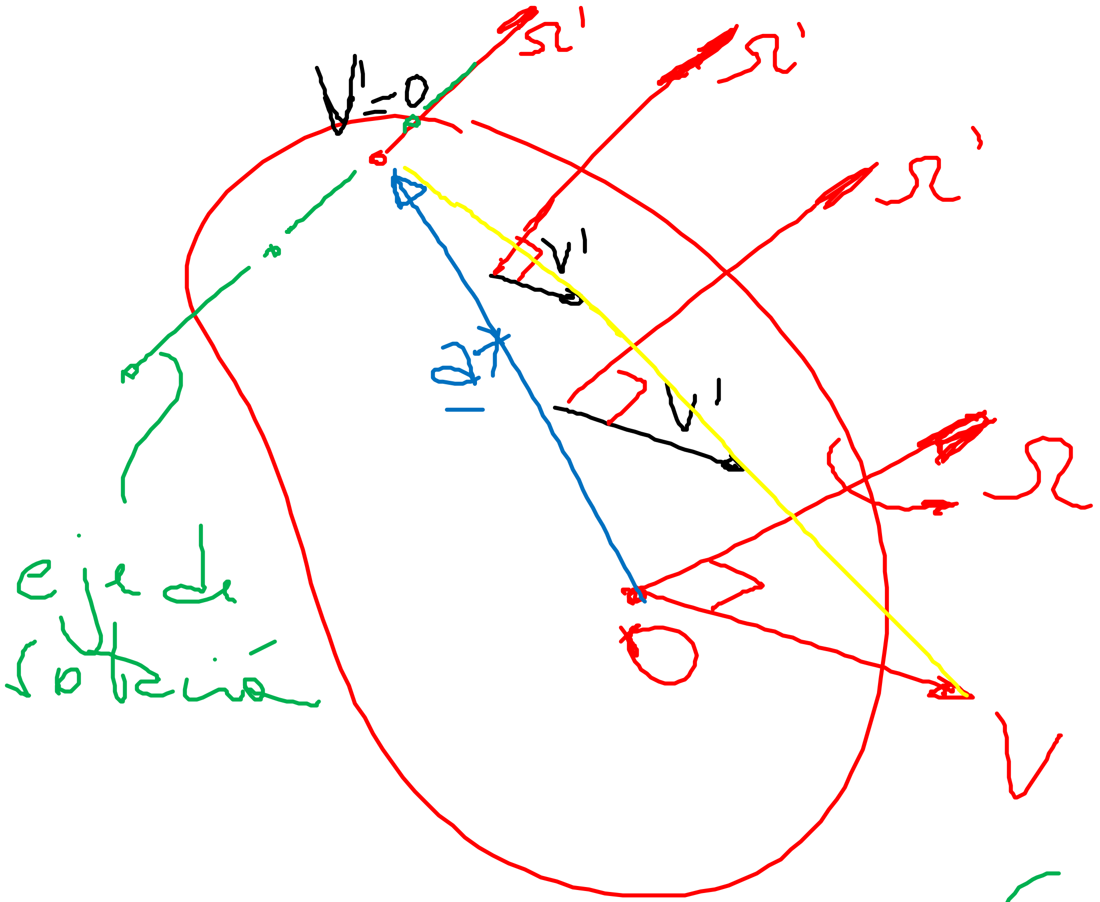
$$\underset{=}{\tilde{\Omega}}^2 = \underline{\Omega} \otimes \underline{\Omega} - \Omega^2 \underset{=}{I}$$

$$\begin{array}{ccc|cc} & & & 0 & \Omega_3 \\ & & & -\Omega_3 & 0 \\ & & & \Omega_2 & -\Omega_1 \\ \hline 0 & \Omega_3 & -\Omega_2 & -\Omega_3^2-\Omega_2^2 & \Omega_1\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 & \Omega_1\Omega_2 & -\Omega_3^2-\Omega_1^2 \\ \Omega_2 & -\Omega_1 & 0 & \Omega_1\Omega_3 & \end{array}$$

$$\begin{array}{cc|c} & & \tilde{\Omega} \\ & & \Omega^T \\ \hline -\tilde{\Omega} & \Omega & \Omega^2 I - \cancel{\Omega \otimes \Omega} + \cancel{\Omega \otimes \Omega} \end{array}$$

$$\underline{a} = \frac{1}{\Omega^2}\left[-\underset{=}{\tilde{\Omega}} \quad \Omega \right] \begin{pmatrix} -V \\ 0 \end{pmatrix} =$$

$$\underline{a} = \frac{\Omega \times V}{\Omega^2}$$



i) Todo V' ⊥ Ω

$$\underline{a}^* = \frac{\underline{\Omega} \times \underline{V}}{\Omega^2}$$

$$\underline{a} = \underline{a}^* + \alpha \underline{\Omega}$$

α arbitrario

Cualquier pto sobre el eje de rotación tiene velocidad $\underline{V}' = \underline{0}$

$$T_{rot} = \frac{1}{2}\sum_i m\left(\Omega^2 r^2 - (\underline{\Omega}\cdot\underline{r})^2\right) =$$

$$= \frac{1}{2}\underline{\Omega}\cdot\sum m r^2\,\underline{\underline{I}}\;\underline{\Omega} - \frac{1}{2}\underline{\Omega}\cdot\sum_i m\,\underline{r}\otimes\underline{r}\;\underline{\Omega}$$

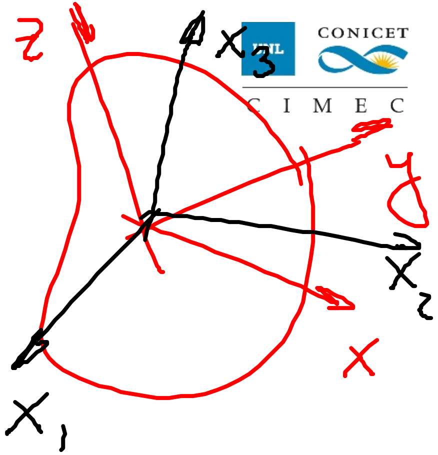
$$= \frac{1}{2}\underline{\Omega}\cdot\underbrace{\left[\sum_i m r^2\,\underline{\underline{I}} - m\,\underline{r}\otimes\underline{r}\right]}_{\underline{\underline{I}}}\underline{\Omega}$$

tensor de inercia del CR

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\underline{\Omega}\cdot\underline{\underline{J}}\;\underline{\Omega}$$

$\underline{\underline{J}}$ es un tensor simétrico

$$\underline{\underline{J}} = \begin{bmatrix} \sum m(y^2+z^2) & & SIM \\ -\sum m\, y x & \sum m(x^2+z^2) & \\ -\sum m\, z x & -\sum m\, z y & \sum m(x^2+y^2) \end{bmatrix}$$



J_{xx}, J_{yy}, J_{zz} mom de inercia resp de ejes x, y, z

TENSOR ADITIVO

Como todo tensor, puede ser diagonalizado.

En los ejes x_1, x_2, x_3 $\Longrightarrow$
$$\underline{\underline{J}} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

"Ejes ppales de inercia"

En los ejes ppales de inercia:

$$T_{rot} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) = \frac{1}{2}\,\underline{\Omega}\cdot I\,\underline{\Omega}$$

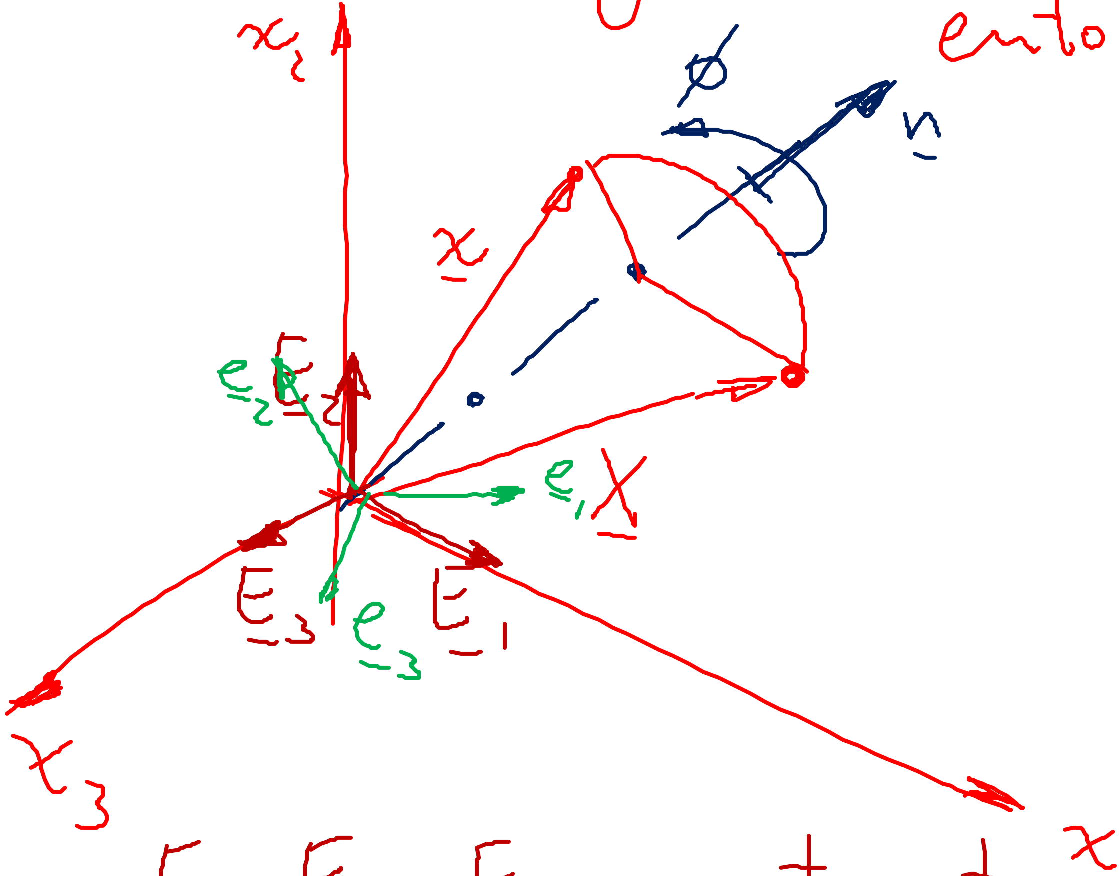
CR no tiene simetrías ⇒ trompo asimétrico.

CR c/2 momentos ppales iguales ($I_1 = I_2$) ⇒ " simétrico.

CR c/3 momentos ppales iguales $I_1 = I_2 = I_3$ ⇒ " esférico

CINEMATICA DE ROTACIONES FINITAS

Movimiento esférico: corresponde a la rotación de un CR entorno a un punto fijo en el espacio



$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$ posición de P en la config referencia

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$ posición actual de P.

$\underline{E}_1, \underline{E}_2, \underline{E}_3$ vect ortonormales fijos al cuerpo.

$\underline{e}_1, \underline{e}_2, \underline{e}_3$ los mismos vect ortonormales luego de la transformación

Mov CR

i) la long del vector posición de P debe ser cte

ii) el áng relativo entre 2 vect arbitrarios fijos al cuerpo debe ser cte.

La transf de $X \rightarrow x$ puede ser expresada por una transf lineal

$$\underline{x} = \underline{\underline{R}}\,\underline{X}$$

de la misma manera

$$\underline{e}_i = \underline{\underline{R}}\,\underline{E}_i$$

Como la long del radio vect es cte:

$$\underline{x}^T \underline{x} = \underline{X}^T \underline{X} = \underline{X}^T \underline{\underline{R}}^T \underline{\underline{R}}\, \underline{X} \Rightarrow \underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{I}}$$

$\underline{\underline{R}}$ es ortogonal $\Rightarrow \underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$

Llamado

$$\underline{\underline{A}} = [\underline{E}_1, \underline{E}_2\, \underline{E}_3] \qquad \underline{\underline{B}} = [\underline{e}_1, \underline{e}_2\, \underline{e}_3]$$

$$\underline{\underline{B}} = \underline{\underline{R}}\, \underline{\underline{A}}$$

Siendo $\underline{E}_i$ unitarios terna dextrógira $\Rightarrow$ det $\underline{\underline{A}} = 1$

det $\underline{\underline{B}} = 1$ $\Rightarrow$ det $\underline{\underline{R}} = 1$

$\underline{\underline{R}}$: matriz ortog propia

Autovalores de $\underline{\underline{R}}$

$$\det \underline{\underline{R}} = \lambda_1 \lambda_2 \lambda_3 = 1$$

Se puede verificar que λ_i tiene la forma:

$$\lambda_1 = 1$$

$$\lambda_{2,3} = \exp(\pm i\phi) = \cos\phi \pm i \,\mathrm{sen}\,\phi$$

Luego, $\underline{\underline{R}}$ admite al menos un autovector $\underline{n}$

$$\underline{\underline{R}}\,\underline{n} = \underline{n} \qquad \underline{n}: \text{eje de rotación!}$$

Los autovectores asociados a $\lambda_{2,3}$ serán complejos conjugados

Expresamos los autovectores de $\underline{\underline{R}}$ en una matriz $\underline{\underline{X}}$:

$$\underline{\underline{X}} = \left[\underline{n} \quad \underline{u}+i\underline{v} \quad \underline{u}-i\underline{v} \right]$$

El prob de autovalores, en forma matricial, se expresa:

$$\underline{\underline{R}}\,\underline{\underline{X}} = \underline{\underline{X}}\,\underline{\underline{\Lambda}}$$

$$\underline{\underline{R}}\,\underline{\underline{X}}\,\underline{\underline{X}}^{*} = \underline{\underline{R}} = \underline{\underline{X}}\,\underline{\underline{\Lambda}}\,\underline{\underline{X}}^{*}$$

$$\underline{\underline{X}}^{-1} = \underline{\underline{X}}^{*} \left(\text{transp conjugado}\right)$$

$$\underline{\underline{X}}^{*} = \begin{bmatrix} \underline{n}^T \\ (\underline{U} - i\underline{V})^T \\ (\underline{U} + i\underline{V})^T \end{bmatrix}$$

$$\underline{\underline{X}}^{*}\underline{\underline{X}} = \begin{bmatrix} \underline{n}^T\underline{n} & \underline{n}^T(\underline{u} + i\underline{v}) & \vdots \\ \vdots & (\underline{U} - i\underline{v})^T(\underline{U} + i\underline{V}) & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} = \underline{\underline{I}}$$

$$\underline{n}^T\underline{u} = \underline{n}^T\underline{v} = 0$$

$$\underline{U}^T\underline{V} = 0$$

$$U^TU + V^TV = 1$$

$\underline{u}, \underline{v}$ forman una dupla de vect ortog en un plano normal a $\underline{n}$

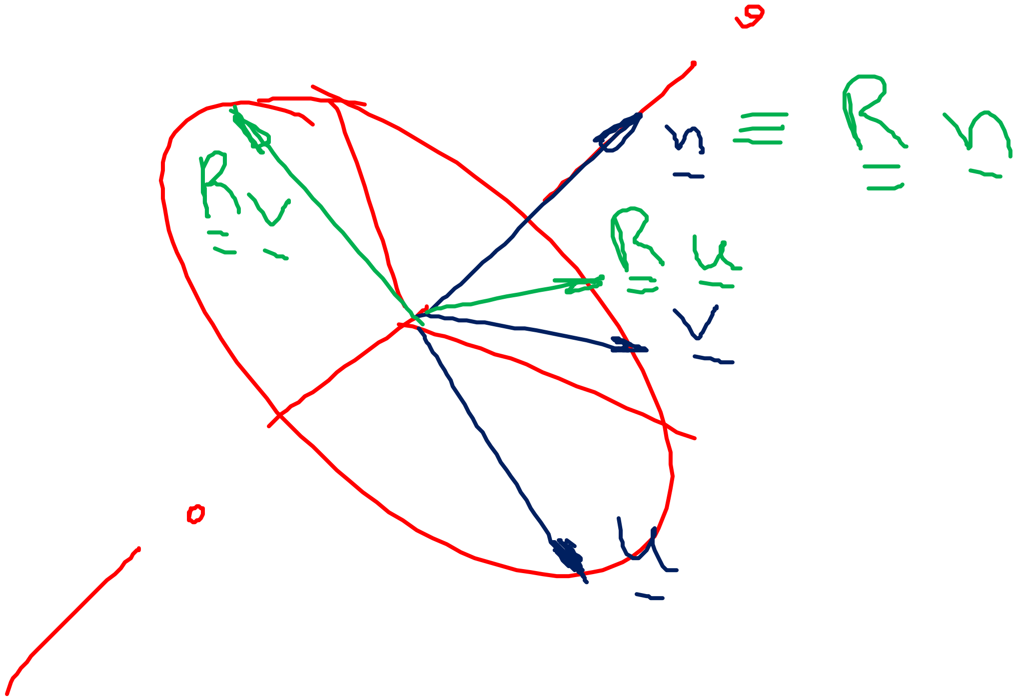
$$\underline{\underline{R}}(\underline{u}+i\underline{v}) = \exp(i\phi)\,(\underline{u}+i\underline{v})$$

Desarrollando:

$$\underline{\underline{R}}\,\underline{u} = \underline{u}\cos\phi - \underline{v}\sin\phi$$

$$\underline{\underline{R}}\,\underline{v} = \underline{u}\sin\phi + \underline{v}\cos\phi$$

O sea, $\forall\ \underline{u}, \underline{v}$ sufre una rotación plana de ángulo ϕ en un plano perpendicular a $\underline{n}$.



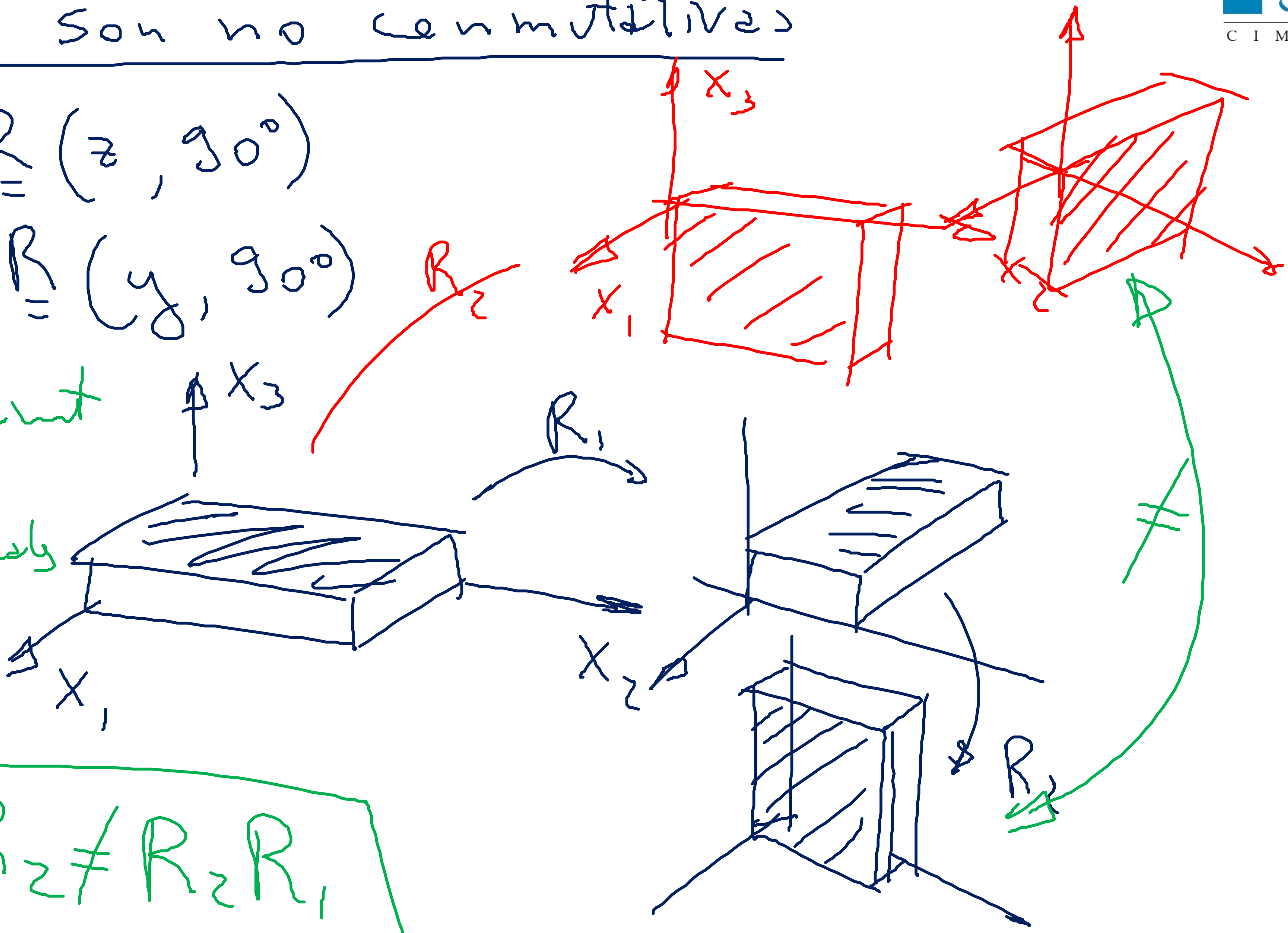
$\underline{n} \equiv \underline{\underline{R}}\,\underline{n}$

Rotaciones son no conmutativas

$$\underline{\underline{R}}_1 = \underline{\underline{R}}(z, 90^\circ)$$

$$\underline{\underline{R}}_2 = \underline{\underline{R}}(y, 90^\circ)$$

Se verif conmut en el caso de rot infinitesimales



$$\boxed{R_1 R_2 \neq R_2 R_1}$$

Expresión del tensor de rotación

$$\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{I}} \Longrightarrow$$ 6 ecs de restricción

$$\underline{\underline{R}} = \left[\underline{r}_1 \; \underline{r}_2 \; \underline{r}_3\right] \Longrightarrow r_i^T r_j = \delta_{ij}$$

Luego $\underline{\underline{R}} = \underline{\underline{R}}(\alpha_1, \alpha_2, \alpha_3)$ función de 3 parámetros

Operador de proyección

$$\underline{\underline{P}}_n = \tilde{\underline{\underline{n}}}^T \tilde{\underline{\underline{n}}} = \underline{\underline{I}} - \underline{n}\,\underline{n}^T$$

$$\underline{\underline{P}}_n \underline{v} = \underline{v} - \underline{n}\left(\underline{n}^T \underline{v}\right) =$$

$$=$$

$$\underline{n}^T \underline{n} = 1$$

Tensor spin

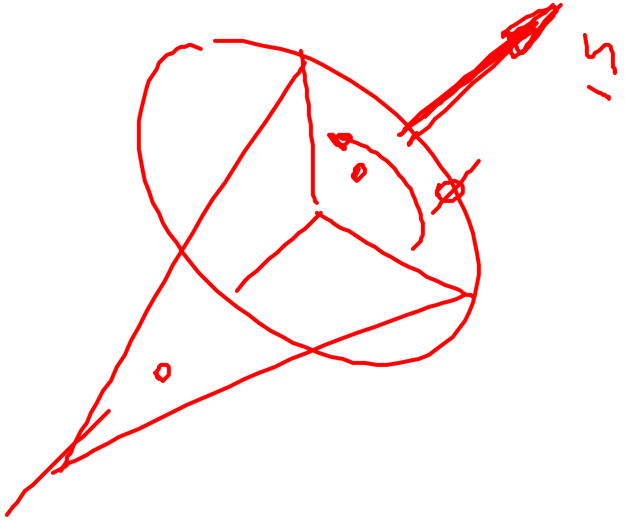
$$\tilde{\underline{\underline{n}}} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} = spin(\underline{n})$$

$$\tilde{\underline{\underline{n}}}\,\underline{v} = \underline{n} \times \underline{v}$$

$$\tilde{\underline{\underline{n}}}\,\underline{n} = \underline{0}$$

$$\begin{array}{ccc|cc}
 & & & 0 & -n_3 \\
 & & & n_3 & 0 \\
 & & & -n_2 & n_1 \\
\hline
0 & n_3 & -n_2 & n_3^2+n_2^2 & \\
-n_3 & 0 & n_1 & -n_1 n_2 & n_3^2+n_1^2 \\
n_2 & -n_1 & 0 & -n_1 n_3 &
\end{array}$$

$$\underline{\underline{R}} = \underline{\underline{I}} \cos\phi + (1-\cos\phi)\, \underline{n}\,\underline{n}^T + \underline{\underline{\tilde{n}}} \sin\phi \quad \Big| \quad \left(= \exp(\underline{\underline{\tilde{n}}}\,\phi) \right)$$



$$\underline{x} = \underline{\underline{R}}\,\underline{X}$$

$\underline{n}, \phi$

pero $\underline{n}^T \underline{n} = 1$

3 parámetros libres

Mapeo exponencial

$$\underline{x} = \underline{\underline{R}}\,\underline{X}$$

Diferenciamos respecto de ϕ

$$\frac{d\underline{x}}{d\phi} = \frac{d\underline{\underline{R}}}{d\phi}\,\underline{X} = \frac{d\underline{\underline{R}}}{d\phi}\,\underline{\underline{R}}^T\,\underline{x}$$

$$\frac{d\underline{\underline{R}}}{d\phi} = \underline{\underline{\tilde{n}}}\cos\phi - (\underline{\underline{I}} - \underline{n}\,\underline{n}^T)\sin\phi$$

Se verifica facilmente que $\dfrac{d\underline{\underline{R}}}{d\phi}\,\underline{\underline{R}}^T = \underline{\underline{\tilde{n}}}$

$$\boxed{\text{EJERCICIO}}$$

Luego

$$\begin{cases} \dfrac{d\underline{x}}{d\phi} - \underline{\underline{\tilde{n}}}\,\underline{x} = 0 \\ \underline{x}(0) = \underline{X} \end{cases}$$

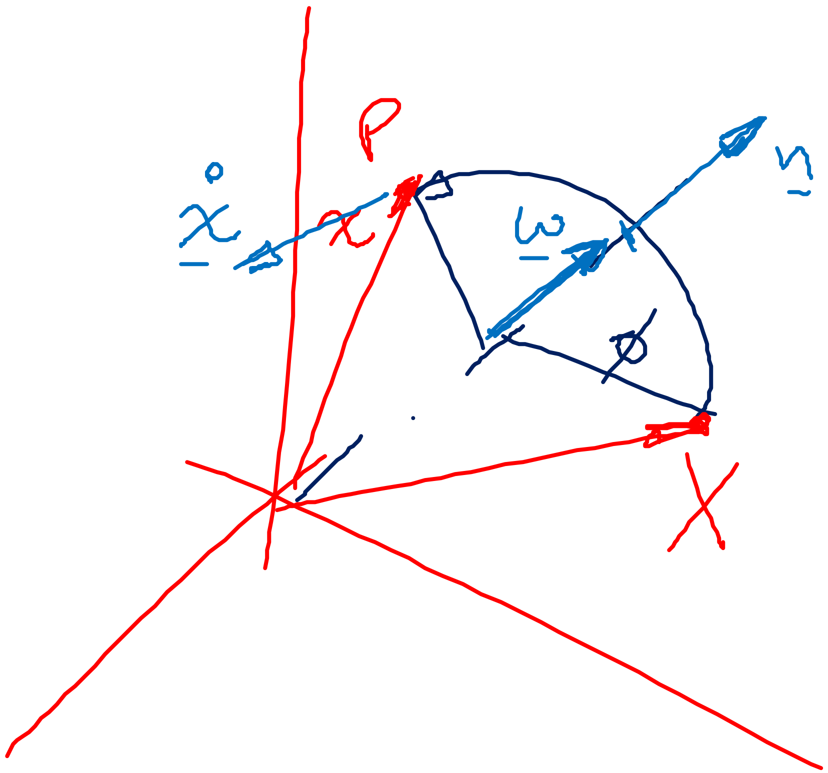
Ec dif en ϕ

La solución de esta EDO

$$\underline{x} = \exp(\underline{\underline{\tilde{n}}}\,\phi)\,\underline{X}$$

$$\therefore \quad \underline{\underline{R}} = \exp(\underline{\underline{\tilde{n}}}\,\phi)$$

Análisis de velocidad p/ mov esférico



$$x = RX \qquad R^T = R^{-1}$$

$$v_p = \dot{\underline{x}} = \dot{\underline{\underline{R}}}\,\underline{X} = \underbrace{\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T}\,\underline{x}$$

$$\frac{d}{dt}\big(\underbrace{\underline{\underline{R}}\,\underline{\underline{R}}^T}_{I}\big) = \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T + \underline{\underline{R}}\,\dot{\underline{\underline{R}}}^T = 0$$

$$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

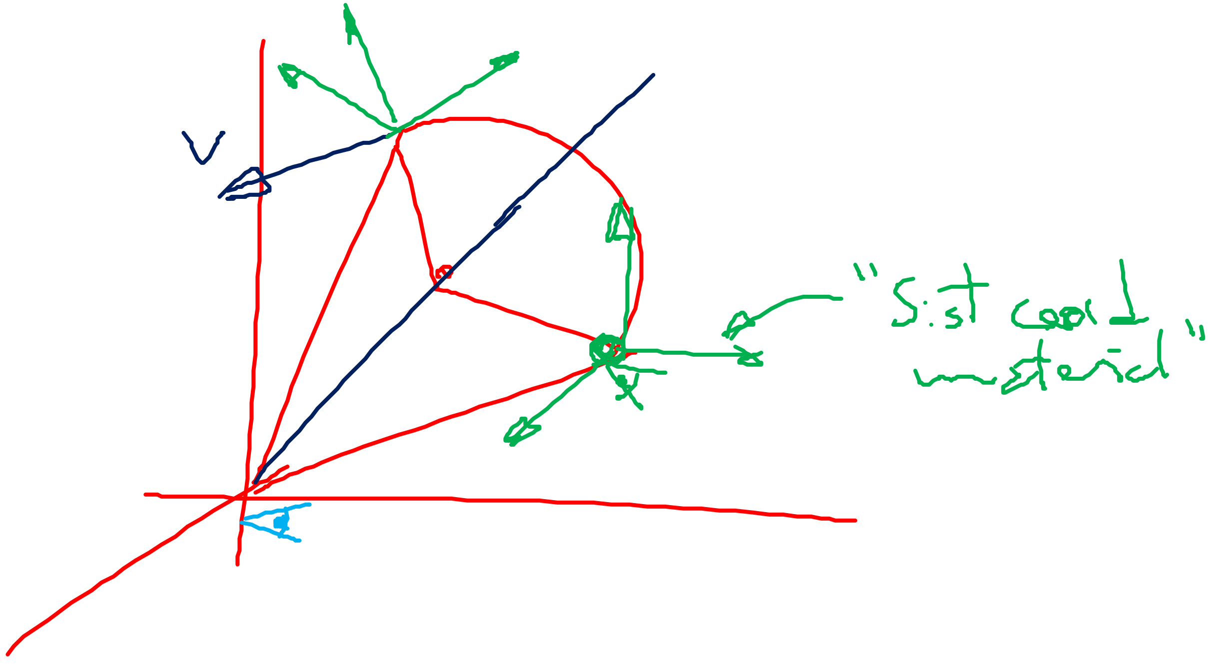
$\therefore$ $\dot{\underline{R}}\,\underline{R}^T$ antisimétrica

Definimos $\tilde{\underline{\underline{\omega}}} = \dot{\underline{R}}\,\underline{R}^T$ Matriz de velocidades angulares expresada en coord espaciales

$$\dot{\underline{x}} = \tilde{\underline{\underline{\omega}}}\,\underline{x} = \underline{\omega} \times \underline{x}$$

$$\underline{\omega} = \text{vect}\left(\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T\right)$$

La vel del punto P la puedo expresar en coord Materiales.



$$V_p = R^T v_p = \underbrace{R^T \dot{R}}_{A\,S} X$$

Definimos: $\tilde{\underline{\underline{\Omega}}} = \underline{\underline{R}}^T \dot{\underline{\underline{R}}}$ "matriz de velocidad angular material"

$$\underline{\Omega} = \text{vect}\left(\underline{\underline{R}}^T \dot{\underline{\underline{R}}}\right)$$

$$\boxed{\begin{array}{l} \dot{\underline{\underline{R}}} = \tilde{\underline{\underline{\omega}}}\,\underline{\underline{R}} = \underline{\underline{R}}\,\tilde{\underline{\underline{\Omega}}} \\ \tilde{\underline{\underline{\omega}}} = \underline{\underline{R}}\,\tilde{\underline{\underline{\Omega}}}\,\underline{\underline{R}}^T \qquad \tilde{\underline{\underline{\Omega}}} = \underline{\underline{R}}^T\,\tilde{\underline{\underline{\omega}}}\,\underline{\underline{R}} \end{array}}$$

$$\tilde{\underline{\underline{\omega}}} = \underline{\underline{R}}\,\tilde{\underline{\underline{\Omega}}}\,\underline{\underline{R}}^T = \widetilde{\left(\underline{\underline{R}}\,\underline{\Omega}\right)}$$

$$\underline{\omega} = \underline{\underline{R}}\,\underline{\Omega} \quad\Longrightarrow\quad \underline{\Omega} = \underline{\underline{R}}^T\,\underline{\omega}$$

$$\tilde{\underline{\underline{\omega}}}\,\underline{a} = \underline{\underline{R}}\,\tilde{\underline{\underline{\Omega}}}\left(\underline{\underline{R}}^T\underline{a}\right) = \underline{\underline{R}}\left(\underline{\Omega}\times\underline{\underline{R}}^T\underline{a}\right) =$$

$$= \underline{\underline{R}}\,\underline{\Omega}\times\left(\underline{\underline{R}}\,\underline{\underline{R}}^T\underline{a}\right) = \left(\underline{\underline{R}}\,\underline{\Omega}\right)\times\underline{a} = \widetilde{\left(\underline{\underline{R}}\,\underline{\Omega}\right)}\,\underline{a}$$

$\underline{a}$ arbitrario $\Rightarrow$ $\tilde{\underline{\underline{\omega}}} = \widetilde{\left(\underline{\underline{R}}\,\underline{\Omega}\right)}$ $\Rightarrow$ $\omega = \underline{\underline{R}}\,\underline{\Omega}$

Expresión explícita de la velocidad angular

$\underline{\omega}, \underline{\Omega}$ función de $\underline{n}, \phi$ y de $\dot{\underline{n}}, \dot{\phi}$

$$\underline{\underline{R}}\,\underline{n} = \underline{n} \qquad \underline{\underline{R}}^T \underline{n} = \underline{n}$$

$$\underline{\underline{R}}^T \left(\underline{\underline{R}} - \underline{\underline{I}}\right) \dot{\underline{n}} = \overset{R^T}{-}\dot{\underline{\underline{R}}}\,\underline{n} \qquad \left(\underline{\underline{R}}^T - \underline{\underline{I}}\right) \dot{\underline{n}} = -\dot{\underline{\underline{R}}}^T \underline{n}$$

$$\left(\underline{\underline{R}}^T - \underline{\underline{I}}\right) \dot{\underline{n}} = \underline{\underline{R}}^T \dot{\underline{\underline{R}}} = \tilde{\underline{\underline{\Omega}}}\,\underline{n} \qquad \left(\underline{\underline{R}} - \underline{\underline{I}}\right) \dot{\underline{n}} = \tilde{\underline{\underline{\omega}}}\,\underline{n}$$

$$\tilde{\underline{\underline{n}}}\,\underline{\Omega} = \left(\underline{\underline{I}} - \underline{\underline{R}}^T\right) \dot{\underline{n}} \qquad \tilde{\underline{\underline{n}}}\,\underline{\omega} = \left(\underline{\underline{R}} - \underline{\underline{I}}\right) \dot{\underline{n}}$$

Ver en el libro $\Longrightarrow$ $\underline{n}^T \underline{\Omega} = \dot{\phi}$ $\quad$ $\underline{n}^T \underline{\omega} = \dot{\phi}$

Luego:

$$\begin{bmatrix} \tilde{\underline{\underline{n}}} \\ \underline{n}^T \end{bmatrix} \underline{\Omega} = \begin{pmatrix} (\underline{\underline{I}} - \underline{\underline{R}}^T)\,\dot{\underline{n}} \\ \dot{\phi} \end{pmatrix} \qquad \begin{bmatrix} \tilde{\underline{\underline{n}}} \\ \underline{n}^T \end{bmatrix} \underline{\omega} = \begin{pmatrix} (\underline{\underline{R}} - \underline{\underline{I}})\,\dot{n} \\ \dot{\phi} \end{pmatrix}$$

Resolviendo:

$$\underline{\omega} = \overbrace{-\tilde{\underline{\underline{n}}}(\underline{\underline{R}} - \underline{\underline{I}})}^{M^T}\,\dot{\underline{n}} + \underline{n}\dot{\phi} = \underline{\underline{M}}\,\dot{\underline{n}} + \underline{n}\dot{\phi}$$

$$\underline{\Omega} = \underbrace{-\tilde{n}(\underline{\underline{I}} - \underline{\underline{R}}^T)}_{M}\,\dot{\underline{n}} + \underline{n}\dot{\phi} = \underline{\underline{M}}^T\dot{\underline{n}} + \underline{n}\dot{\phi}$$

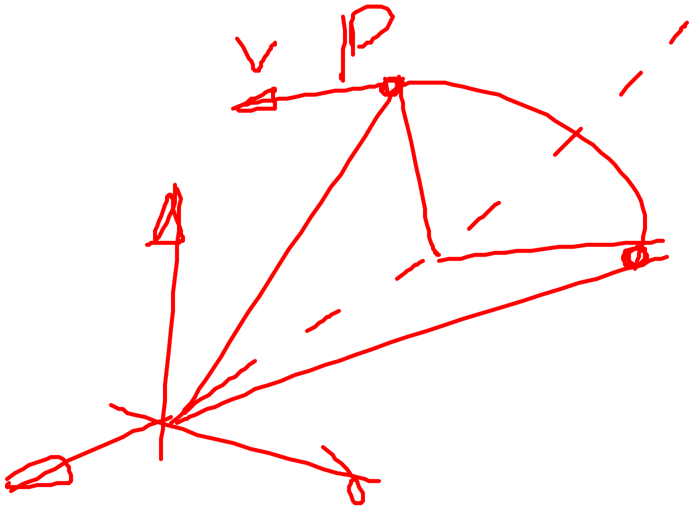
$$\underline{\underline{M}} = \sin\phi\, \underline{\underline{I}} - (1 - \cos\phi)\tilde{\underline{\underline{n}}}$$

$$M = -\underline{\underline{\tilde{n}}}\left(\underline{\underline{I}} - \underline{\underline{R}}^T\right)$$

$$M^T = -\left(\underline{\underline{R}} - \underline{\underline{I}}\right)\left(\underline{\underline{\tilde{n}}}\right)^T = +\left(\underline{\underline{R}} - \underline{\underline{I}}\right)\underline{\underline{\tilde{n}}} = \underline{\underline{\tilde{n}}}\left(\underline{\underline{R}} - \underline{\underline{I}}\right)$$

$$\underline{\underline{R}}\,\underline{\underline{\tilde{n}}} = \underline{\underline{R}}\,\underline{\underline{\tilde{n}}}\,\underline{\underline{R}}^T\underline{\underline{R}} = \widetilde{\left(\underline{\underline{R}}\,\underline{n}\right)}\,\underline{\underline{R}} = \underline{\underline{\tilde{n}}}\,\underline{\underline{R}}$$

Aceleraciones angulares



$$\underline{a}_p = \dot{\underline{v}}_p = \ddot{\underline{x}}_p = \ddot{\underline{\underline{R}}}\,\underline{X} =$$

$$= \underbrace{\ddot{\underline{\underline{R}}}\,\underline{\underline{R}}^T}_{\text{matriz de aceleraciones ang}} \underline{x}$$

AS

SIMET

$$\underline{\underline{\alpha}} = \ddot{\underline{\underline{R}}}\,\underline{\underline{R}}^T = \frac{d}{dt}\Big(\underbrace{\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T}_{\underline{\underline{\tilde{\omega}}}}\Big) - \underbrace{\dot{\underline{\underline{R}}}\,\dot{\underline{\underline{R}}}^T}_{\underbrace{\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T\underline{\underline{R}}\,\dot{\underline{\underline{R}}}^T}_{\underline{\underline{\tilde{\omega}}}\,\underline{\underline{\tilde{\omega}}}^T}} = \dot{\underline{\underline{\tilde{\omega}}}} + \underline{\underline{\tilde{\omega}}}\,\underline{\underline{\tilde{\omega}}}$$

tensor derivado de la vel angular

"aceleración centrífuga"

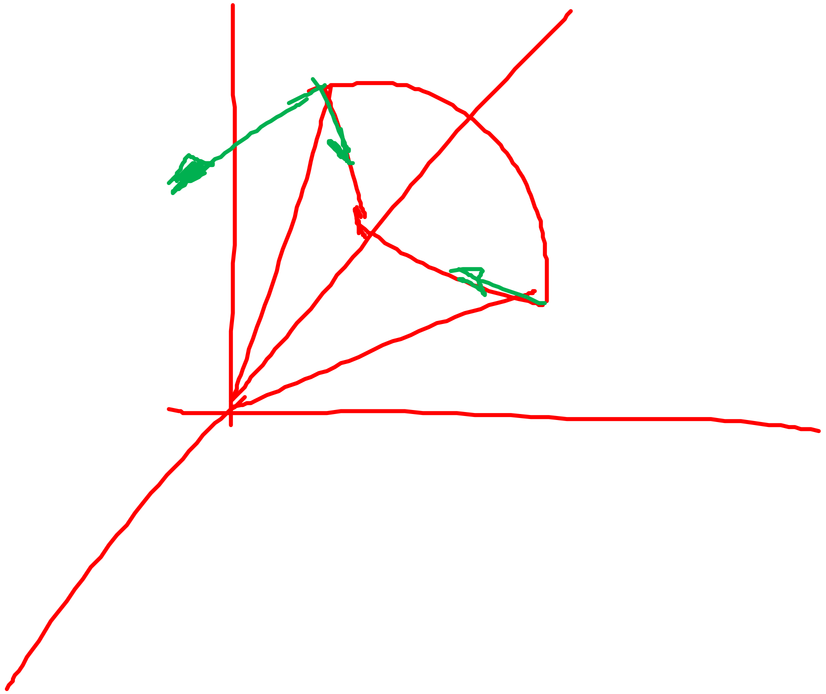
$$\underline{a}_p = \dot{\underline{\underline{\tilde{\omega}}}}\,\underline{x} + \underbrace{\underline{\underline{\tilde{\omega}}}\,\underline{\underline{\tilde{\omega}}}\,\underline{x}}_{\text{radial hacia el eje}}$$

En coord materiales:

acel del pto $\underline{A}_p = \underline{\underline{R}}^T \underline{a}_p = \underbrace{\underline{\underline{R}}^T \ddot{\underline{\underline{R}}}} \underline{X}$

acel angular $A = \underline{\underline{R}}^T \ddot{\underline{\underline{R}}} = \frac{d}{dt}\left(\underline{\underline{R}}^T \dot{\underline{\underline{R}}}\right) - \dot{\underline{\underline{R}}}^T \dot{\underline{\underline{R}}} = \dot{\tilde{\underline{\underline{\Omega}}}} + \tilde{\underline{\underline{\Omega}}} \tilde{\underline{\underline{\Omega}}}$

$\underline{A} = \underline{\underline{R}}^T \underline{\alpha}$



Movimiento de rotación infinitesimal

$$\underline{x} = \underline{\underline{R}}\,\underline{X} \qquad \overset{\circ}{\underline{x}} = \overset{\circ}{\underline{\underline{R}}}\,\underline{X}$$

$$\delta\underline{x} = \delta\underline{\underline{R}}\,\underline{X}$$ desplaz. virtuales

$$= \underbrace{\delta\underline{\underline{R}}\,\underline{\underline{R}}^T}\,\underline{x} = \delta\underline{\underline{\tilde{\theta}}}\,\underline{x}$$

$$\delta\underline{x} = \underline{\underline{R}}\,\underline{\underline{R}}^T\delta\underline{\underline{R}}\,\underline{X} = \underline{\underline{R}}\,\delta\underline{\underline{\tilde{\Theta}}}\,\underline{X}$$

$$\delta\underline{\underline{\tilde{\theta}}} = \delta\underline{\underline{R}}\,\underline{\underline{R}}^T$$

$$\delta\underline{\underline{\hat{\Theta}}} = \underline{\underline{R}}^T\delta\underline{\underline{R}} \Rightarrow \delta\underline{\underline{R}} = \delta\underline{\underline{\tilde{\theta}}}\,\underline{\underline{R}} = \underline{\underline{R}}\,\delta\underline{\underline{\tilde{\Theta}}}$$

$$\underline{\underline{R}}\,\delta\underline{\Theta} = \delta\underline{\theta}$$

Variación de velocidad angular

$$\tilde{\underline{\underline{\Omega}}} = \underline{\underline{R}}^T \dot{\underline{\underline{R}}} \Rightarrow \delta\tilde{\underline{\underline{\Omega}}} = \delta\underline{\underline{R}}^T \dot{\underline{\underline{R}}} + \underline{\underline{R}}^T \delta\dot{\underline{\underline{R}}}$$

$$\delta\tilde{\underline{\underline{\Theta}}} = \underline{\underline{R}}^T \delta\underline{\underline{R}} \Rightarrow \frac{d}{dt}\left(\delta\tilde{\underline{\underline{\Theta}}}\right) = \dot{\underline{\underline{R}}}^T \delta\underline{\underline{R}} + \underline{\underline{R}}^T \delta\dot{\underline{\underline{R}}}$$

Luego:

$$\delta\tilde{\underline{\underline{\Omega}}} = \delta\dot{\tilde{\underline{\underline{\Theta}}}} + \delta\underline{\underline{R}}^T \dot{\underline{\underline{R}}} - \dot{\underline{\underline{R}}}^T \delta\underline{\underline{R}} = \delta\dot{\tilde{\underline{\underline{\Theta}}}} - \delta\tilde{\Theta}\tilde{\Omega} + \tilde{\Omega}\delta\tilde{\Theta}$$

$$-\delta\tilde{\Theta}\underbrace{R^T \dot{R}}_{\tilde{\Omega}} \qquad \underbrace{\dot{R}^T R}_{-\Omega}\delta\tilde{\Theta}$$

$$\delta\tilde{\underline{\underline{\omega}}} = \delta\dot{\tilde{\Theta}} + \delta\tilde{\Theta}\tilde{\omega} - \tilde{\omega}\delta\tilde{\Theta}$$

$$\delta \underline{\underline{\tilde{\Omega}}} = \delta \underline{\underline{\dot{\tilde{\Theta}}}} \underbrace{- \delta \underline{\underline{\tilde{\Theta}}}\,\underline{\underline{\tilde{\Omega}}} + \underline{\underline{\tilde{\Omega}}}\,\delta \underline{\underline{\tilde{\Theta}}}}$$

$$\left[\tilde{a}\tilde{b} - \tilde{b}\tilde{a}\right] c = a \times (b \times c) - b \times (a \times c) =$$

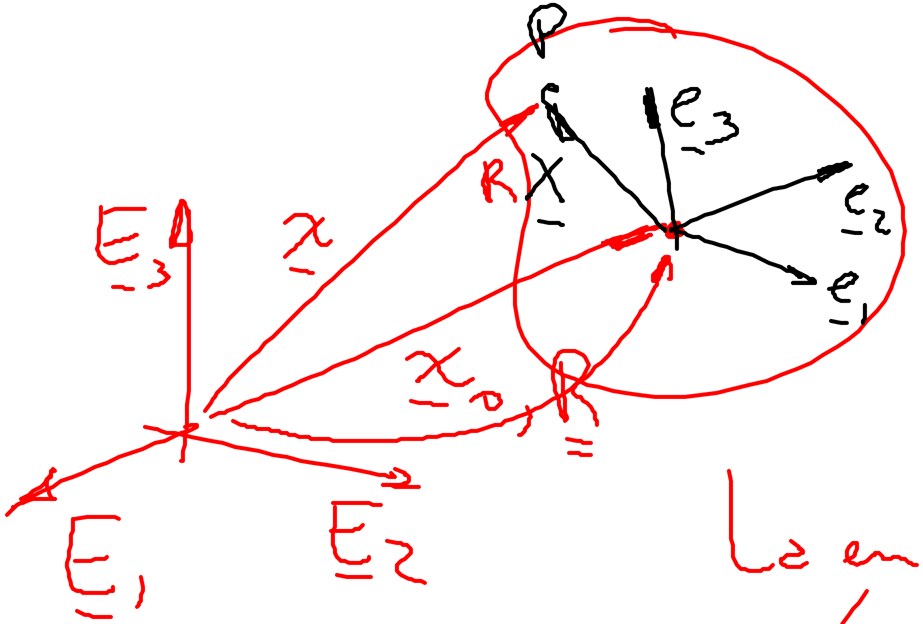
$$= (a \cdot c)\, b - \cancel{(a \cdot b)\, c} - (b \cdot c)\, a + \cancel{(b \cdot a)\, c} =$$

$$= c \times (b \times a) = (a \times b) \times c = \widetilde{(a \times b)}\, c$$

$$\left[\tilde{a}\tilde{b} - \tilde{b}\tilde{a}\right] = \widetilde{a \times b}$$

$$\delta \tilde{\Omega} = \delta \dot{\tilde{\Theta}} + \widetilde{(\Omega \times \delta \Theta)} \Rightarrow \boxed{\begin{array}{l} \delta \Omega = \delta \dot{\Theta} + \tilde{\underline{\Omega}}\, \delta \underline{\Theta} \\ \delta \underline{\omega} = \delta \dot{\underline{\Theta}} - \hat{\underline{\omega}}\, \delta \underline{\Theta} \end{array}}$$

CUERPO RIGIDO



$$\underline{e}_i = \underline{\underline{R}}(t)\,\underline{E}_i$$

(en el instante inicial, $\underline{e}_i \equiv \underline{E}_i$)

$$\underline{x} = \underline{x}_0 + \underline{\underline{R}}(t)\,\underline{X}$$

La energía cinética:

$$K = \frac{1}{2}\left(m\, \mathring{\underline{x}}_0^T \underline{x}_0 + \underline{\Omega}^T \underline{\underline{J}}\, \underline{\Omega} \right)$$

de porque usamos vel ang material

$$K = \frac{1}{2}\int_V \dot{x}^T \dot{x}\, \rho\, dV = \frac{1}{2}\int_V \dot{\underline{x}}_0^T \dot{\underline{x}}_0\, \rho\, dV + \underbrace{\frac{1}{2}\int (\dot{R}X)^T (\dot{R}X)\, \rho\, dV}_{\circledast}$$

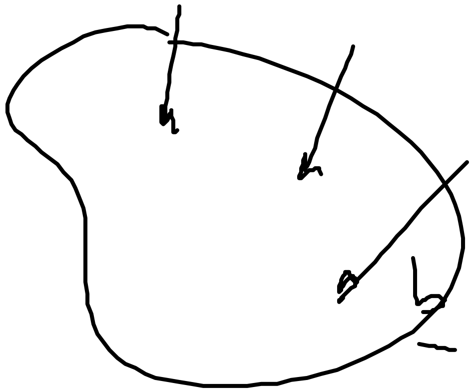
$$\dot{R}X = R\tilde{\Omega}X = -R\tilde{X}\Omega \Rightarrow \tilde{\omega}RX = -\widetilde{RX}\,\omega = -R\tilde{X}R^T\omega$$

$$\circledast = \frac{1}{2}\int + \Omega^T \tilde{X}^T R^T R \tilde{X}\, \Omega\, \rho\, dV = \frac{1}{2}\underline{\Omega}^T \underbrace{\int \tilde{\underline{\underline{X}}}^T \tilde{\underline{\underline{X}}}\, \rho\, dV}_{\underline{\underline{J}}}\, \underline{\Omega}$$

$$K = \frac{1}{2}\left(m\, \dot{\underline{x}}_0^T \dot{\underline{x}}_0 + \underline{\Omega}^T \underline{\underline{J}}\, \underline{\Omega} \right) \qquad I(t) = \underline{\underline{R}}\, \underline{\underline{J}}\, \underline{\underline{R}}^T$$

$$K = \frac{1}{2}\left(m\, \dot{\underline{x}}_0^T \underline{x}_0 + \underline{\omega}^T \underline{\underline{I}}(t)\, \underline{\omega} \right)$$

CARGAS EXTERNAS DE VOLUMEN



$$\mathcal{P} = +\int \underline{x}^T \underline{b}\, dV =$$

$$+\int (x_0 + RX)^T b\, dV$$

$$\therefore \mathcal{P} = +\underline{x}_0^T \underline{f} + \int (RX)^T b\, dV$$

$$\delta\mathcal{P} = +\delta\underline{x}_0^T f + \delta\underline{\theta}^T \underline{C}$$

$$\underline{C} = \int \underline{\underline{\tilde{X}}}\, \underline{R}^T \underline{b}\, dV$$

ECS DE MOVIMIENTO

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0 = \delta \int_{t_1}^{t_2} (K - \mathcal{P})\, dt =$$

$$\delta K = \delta \dot{\underline{x}}_0^T m \dot{\underline{x}}_0 + \delta \underline{\Omega}^T \underline{\underline{J}}\, \underline{\Omega} =$$

$$= \delta \dot{\underline{x}}_0^T m \dot{\underline{x}}_0 + \left(\delta \dot{\underline{\Theta}} + \tilde{\underline{\underline{\Omega}}}\, \delta \underline{\Theta}\right)^T \underline{\underline{J}}\, \underline{\Omega}$$

$$\int_{t_1}^{t_2} \delta K\, dt = \cancel{\delta x_0 m \dot{x}_0}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta \underline{x}_0^T m \ddot{\underline{x}}_0\, dt + \cancel{\delta \underline{\Theta}\, \underline{\underline{J}}\, \Omega}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta \underline{\Theta}^T \underline{\underline{J}}\, \dot{\underline{\Omega}}\, dt - \int_{t_1}^{t_2} \delta \underline{\Theta}^T \tilde{\underline{\underline{\Omega}}}\, \underline{\underline{J}}\, \underline{\Omega}\, dt$$

$$\delta \int_{t_1}^{t} K\,dt = \int_{t_1}^{t_2} \delta \underline{x}_0^T\, m\, \overset{\circ\circ}{\underline{x}}_0\, dt + \int_{t_1}^{t_2} \delta \underline{\Theta}^T \left(\underline{\underline{J}}\, \overset{\circ}{\underline{\Omega}} + \tilde{\underline{\underline{\Omega}}}\, \underline{\underline{J}}\, \underline{\Omega} \right) dt$$

$$\delta \int_{t_1}^{t_2} (-\mathcal{P})\, dt = \int_{t_1}^{t_2} \delta \underline{x}_0^T\, \underline{f} + \delta \underline{\Theta}^T\, \underline{C}$$

$$\begin{cases} m\, \overset{\circ\circ}{\underline{x}}_0 = \underline{f} \\ \underline{\underline{J}}\, \overset{\circ}{\underline{\Omega}} + \tilde{\underline{\underline{\Omega}}}\, \underline{\underline{J}}\, \underline{\Omega} = \underline{C} \end{cases}$$

Pág 114, Landau ⟹ Ecuaciones de Euler

La relación más simple entre componentes del momento angular rotacional y las comp de la vel angular ocurre en un sist de coord móvil con el cuerpo, en aquél P/el cual los ejes coinciden c/los ejes ppales de inercia.

Sea $\frac{d\underline{A}}{dt}$ la tasa de cambio de un vector $\underline{A}$ c/respecto al sist inercial. Si $\underline{A}$ no cambia en un sistema móvil, su tasa de cambio es sólo debida a la rotación y entonces

$$\frac{d\underline{A}}{dt} = \underline{\Omega} \times \underline{A}$$

En un caso gral, tenemos también el cambio de A y resulta:

$$\frac{d\underline{A}}{dt} = \boxed{\frac{d'\underline{A}}{dt}} + \underline{\Omega}\times\underline{A}$$

cambio de $\underline{A}$ en el sist. móvil

Usando estas fórmulas p/ el momento lineal y el angular:

$$\left(\frac{d'\underline{P}}{dt} + \underline{\Omega}\times\underline{P} = \underline{F}\right.$$

$$\left(\frac{d'\underline{M}}{dt} + \underline{\Omega}\times\underline{M} = \underline{C}\right.$$

$$\underline{P} = m\underline{V}$$

$$\underline{M} = J\,\underline{\Omega}$$

$$M_1 = J_1\Omega_1$$
$$M_2 = J_2\Omega_2$$
$$M_3 = J_3\Omega_3$$

$$\mu\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right) = F_1$$

$$\mu\left(\frac{dV_2}{dt} + \Omega_3 V_1 - \Omega_1 V_3\right) = F_2$$

$$\mu\left(\frac{dV_3}{dt} + \Omega_1 V_2 - \Omega_2 V_1\right) = F_3$$

$$J_1 \frac{d\Omega_1}{dt} + \left(J_3 - J_2\right)\Omega_2\Omega_3 = C_1$$

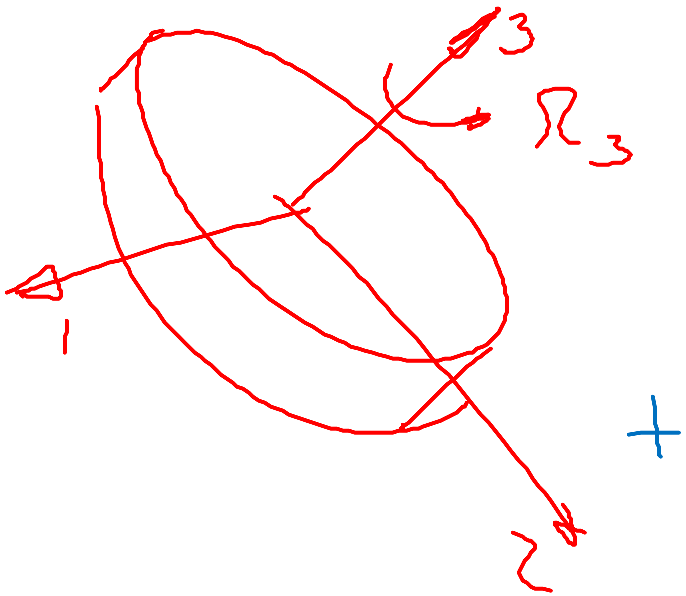
$$J_2 \frac{d\Omega_2}{dt} + \left(J_1 - J_3\right)\Omega_1\Omega_3 = C_2$$

$$J_3 \frac{d\Omega_3}{dt} + \left(J_2 - J_1\right)\Omega_1\Omega_2 = C_3$$

Trompo simétrico:

$$J_1 = J_2$$

$J_1 = J_2 \qquad C_1 = C_2 = C_3 = 0 \Rightarrow \frac{d\Omega_3}{dt} = 0 \qquad \Omega_3 = cte$



$$\frac{d\Omega_1}{dt} + \underbrace{\frac{J_3 - J_2}{J_1}\Omega_2\Omega_3}_{\omega} = 0$$

$$\frac{d\Omega_1}{dt} + \omega\Omega_2 = 0$$

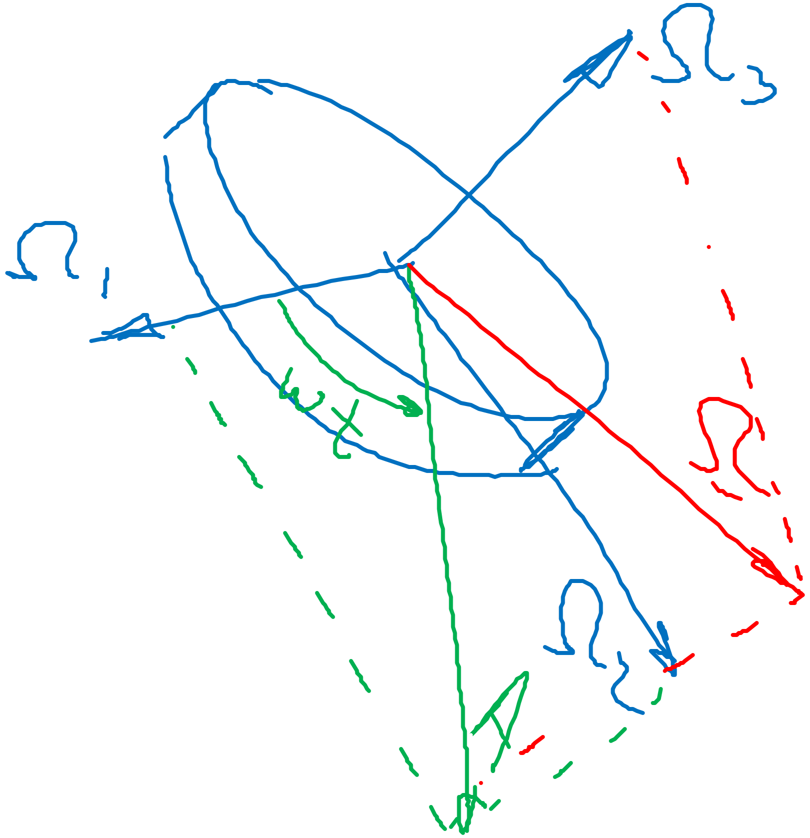
$$+\; i\left(\frac{d\Omega_2}{dt} - \omega\Omega_1\right) = 0$$

$$\frac{d}{dt}\left(\Omega_1 + i\Omega_2\right) = i\omega\left(\Omega_1 + i\Omega_2\right)$$

$$\Omega_1 + i\Omega_2 = A\exp(i\omega t) \qquad A = cte$$

$$\left\{\begin{array}{l}\Omega_1 = A\cos\omega t \\ \Omega_2 = A\sin\omega t \\ \Omega_3 = cte\end{array}\right\} \longrightarrow A = \sqrt{\Omega_1^2 + \Omega_2^2}$$

$$\boxed{\omega = \Omega_3 \, \frac{J_3 - J_1}{J_1}}$$

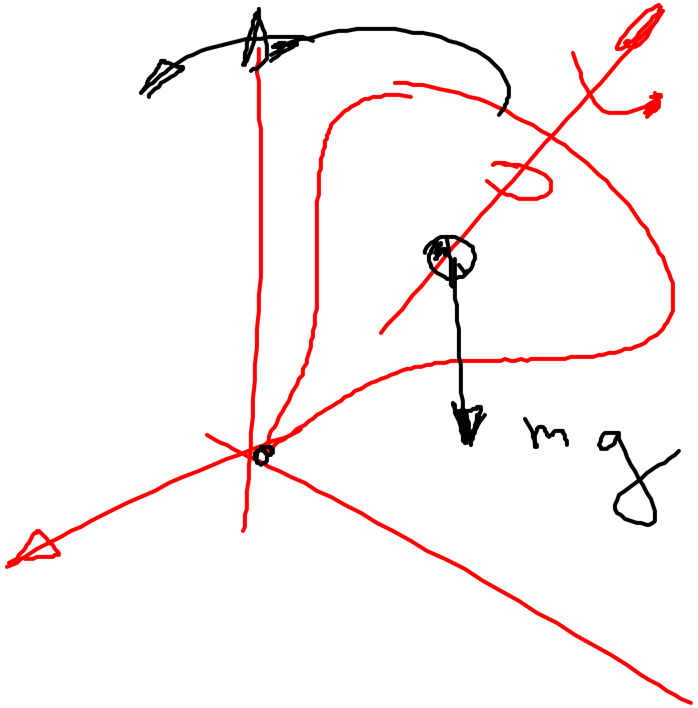


CI ?

i) $\left.\begin{array}{l}\Omega_3 = cte \\ \Omega_1 = \Omega_2 = 0\end{array}\right]$ $\begin{array}{l}\Omega_3 = cte \\ \Omega_1 = \Omega_2 = 0\end{array}$

ii) Ω_1

$\Omega_2 = 0$


O
mg

PARAMETRIZACION DE ROTACIONES

$$\underline{\underline{R}} = \underline{\underline{R}}(\underline{a}) \qquad \underline{a} = (\alpha_1, \alpha_2, \alpha_3)$$

$$\underline{\underline{R}} = \underline{\underline{R}}(\underline{n}, \phi)$$

$$\underline{\omega} = \underline{\underline{P}}(\underline{a})\,\dot{\underline{a}} \qquad \underline{\Omega} = \underline{\underline{Q}}(\underline{a})\,\dot{\underline{a}}$$

$$\underline{\omega} = \underline{\underline{R}}\,\underline{\Omega} \implies \underline{\underline{R}} = \underline{\underline{P}}\,\underline{\underline{Q}}^{-1}$$

i) Vector de rotación Cartesiano (CRV)

$$\underline{\Psi} = \underline{n}\,\phi$$

\# coord mínimo (3)

libre de singularidades

$$\begin{cases} \underline{\underline{R}} = \underline{\underline{I}} + \dfrac{\sin\|\Psi\|}{\|\Psi\|}\,\underline{\underline{\tilde{\Psi}}} + \dfrac{1-\cos\|\Psi\|}{\|\Psi\|^2}\,\underline{\underline{\tilde{\Psi}}}\,\underline{\underline{\tilde{\Psi}}} \\ \underline{\underline{R}} = \exp\left(\underline{\underline{\tilde{\Psi}}}\right) \end{cases}$$

$$\left(\underline{\underline{R}} = \underline{\underline{T}}\,\underline{\underline{T}}^{-1}\right)$$

$$\underline{\Omega} = \underline{\underline{T}}(\underline{\Psi})\,\underline{\dot{\Psi}} \qquad \underline{\omega} = \underline{\underline{T}}^{T}(\underline{\Psi})\,\underline{\dot{\Psi}}$$

$$\underline{\underline{T}}(\underline{\Psi}) = \underline{\underline{I}} + \left(\frac{\cos\Psi - 1}{\Psi^2}\right)\underline{\underline{\tilde{\Psi}}} + \left(1 - \frac{\sin\Psi}{\Psi}\right)\frac{\underline{\underline{\tilde{\Psi}}}\,\underline{\underline{\tilde{\Psi}}}}{\Psi^2}$$

$$\lim_{\Psi \to 0} T(\Psi) = \lim_{\Psi \to 0} R(\Psi) = \underline{\underline{I}}$$

rot "moderadas"

$$R(\Psi) = \exp(\tilde{\Psi}) \simeq \left(I + \tilde{\Psi} + \frac{1}{2}\tilde{\Psi}^2 \right) + \frac{1}{3!}\tilde{\Psi}^3 \ldots$$

$$T(\Psi) = I - \frac{1}{2!}\tilde{\Psi} + \frac{1}{3!}\tilde{\Psi}^2 \ldots$$

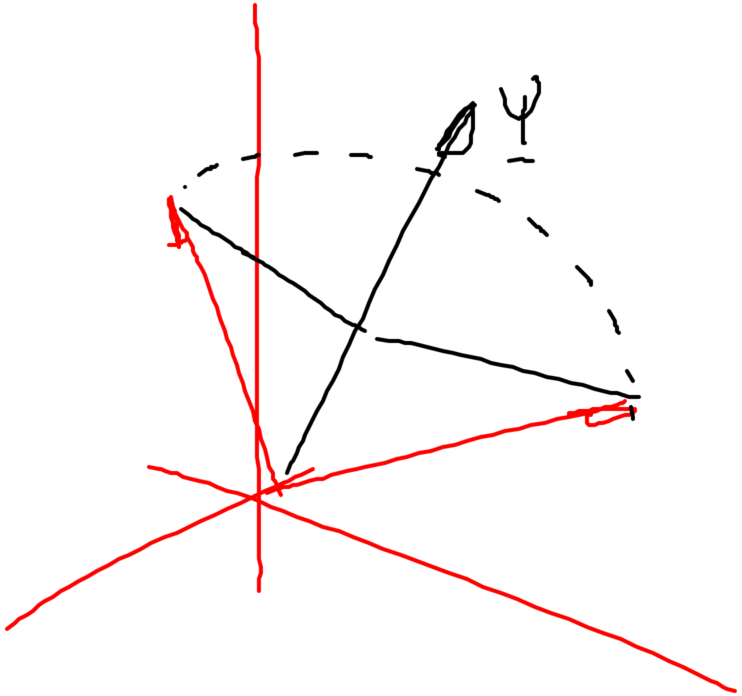
<u>Aceleraciones angulares</u>:

$$\dot{\Omega} = T(\Psi)\ddot{\Psi} + \dot{T}(\Psi)\dot{\Psi}$$

$$\dot{\omega} = T^T(\Psi)\ddot{\Psi} + \dot{T}^T(\Psi)\dot{\Psi}$$

$$\delta\Theta = T(\Psi)\,\delta\Psi$$

$$\delta\theta = T^T(\Psi)\,\delta\Psi$$


ψ

Parámetros de Rodrigues

$$\underline{b} = \underline{n} \tan \phi/2$$

\# parám mínimo (3)

∄ oper trigonométricas

Singularidad en ± π

$$\underline{\underline{R}} = \underline{\underline{I}} + \frac{2}{1+\|b\|^2}\left(\underline{\underline{\tilde{b}}} + \underline{\underline{\tilde{b}}}\,\underline{\underline{\tilde{b}}}\right)$$

$$\underline{\Omega} = \underline{\underline{T}}(b)\,\underline{\dot{b}} \qquad \omega = T^T(b)\,\dot{b}$$

$$T = \frac{2}{1+\|b\|^2}\left(I - \tilde{b}\right)$$

Parámetros de Euler

INCONV # parámetros > 3

vect 4-dimensional

$$\underline{p}^T = [e_0 \ \ \underline{e}^T] = [e_0 \ e_1 \ e_2 \ e_3]$$

$$\left.\begin{array}{l} e_0 = \cos\phi/2 \\ \underline{e} = \underline{n} \sin\phi/2 \end{array}\right\} \Rightarrow e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

$$\underline{\underline{R}} = (2e_0^2 - 1)\,\underline{\underline{I}} + 2\underline{e}\,\underline{e}^T + 2e_0 \underline{\underline{\tilde{e}}}$$

$$\underline{\underline{R}} = \underline{\underline{H}}\,\underline{\underline{G}}^T$$

$$\underline{\underline{H}} = [-\underline{e} \ \ \ e_0\underline{\underline{I}} + \underline{\underline{\tilde{e}}}]$$

$$\underline{\underline{G}} = [-\underline{e} \ \ \ e_0\underline{\underline{I}} - \underline{\underline{\tilde{e}}}]$$

$$\underline{\Omega} = 2\underline{\underline{G}}\,\dot{\underline{p}}$$

$$\underline{\omega} = 2\underline{\underline{H}}\,\dot{\underline{p}}$$

Algebra de quaterniones

Def: quaternion $\longrightarrow$ núm. complejo de 4 dimensiones

$$\hat{q} = q_0 + i q_1 + j q_2 + k q_3 = q_0 + \underline{q}$$

i, j, k : núm imaginarios

$$i^2 = j^2 = k^2 = -1$$

$$jk = -kj = i$$

$$ki = -ik = j$$

$$ij = -ji = k$$

Regla multiplicación

$$\hat{r} = \hat{p}\,\hat{q} = \underbrace{p_0 q_0 - \underline{p}\cdot\underline{q}}_{r_0} + \underbrace{p_0 \underline{q} + q_0 \underline{p} + \tilde{\underline{p}}\,\underline{q}}_{\underline{r}}$$

Nota $\hat{p}\,\hat{q} \neq \hat{q}\,\hat{p}$

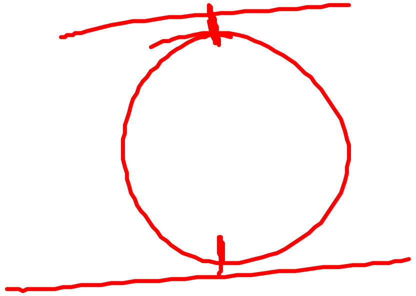
$$\begin{array}{cc|c} & & q_0 \\ & & \underline{q} \\ \hline p_0 & \underline{p}^T & r_0 \\ \underline{p} & p_0 I + \tilde{\underline{p}} & \underline{r} \end{array}$$

Quaternión conjugado

$$\hat{q}^* = q_0 - i q_1 - j q_2 - k q_3 = q_0 - \underline{q}$$

Norma:

$$\|\hat{q}\|^2 = \hat{q}\,\hat{q}^* = q_0^2 + \underline{q}\cdot\underline{q}$$



Quaternión unitario

$$\|\hat{e}\| = 1$$

Relación c/Rotaciones finitas

Sea

$\hat{e} = e_0 + \underline{e}$ quat unitario

$\hat{X} = 0 + \underline{X}$ quat vectorial

Se puede verificar que la rotación finita de $\underline{X}$ a la posición $\underline{x}$ correspondiente a parámetros de Euler $\hat{e} = \cos\alpha + \underline{n}\sin\alpha$

$$\hat{x} = \hat{e}\,\hat{X}\,\hat{e}^*$$

Nota: la rotación inversa

$$\hat{X} = \hat{e}^* \hat{x} \hat{e}$$

$$\hat{x} = \hat{e} \hat{X} \hat{e}^* \hat{e}$$

$$\hat{e}^*$$

$$\hat{e}^* \hat{x} \hat{e} = \underbrace{\hat{e}^* \hat{e}}_{1} \hat{X} \underbrace{\hat{e}^* \hat{e}}_{1}$$

Comp de rotaciones :

$$\hat{x}_1 = \hat{e}_1 \hat{X} \hat{e}_1^*$$

$$\hat{x}_2 = \hat{e}_2 \hat{x}_1 \hat{e}_2^* = \underbrace{\hat{e}_2 \hat{e}_1}_{\hat{e}} \hat{X} \underbrace{\hat{e}_1^* \hat{e}_2^*}_{\hat{e}^*} = \hat{e} \hat{X} \hat{e}^*$$

$$\hat{e} = \hat{e}_2 \hat{e}_1$$

$$\delta \underline{\Theta} \cdot \underline{f}(\underline{\underline{R}}) = 0$$

$$\underline{\underline{R}}(\underline{\Psi})$$

$$\delta \underline{\Psi} \cdot T^T(\Psi) \frac{\partial f}{\partial \Theta} T(\underline{\Psi}) \Delta \Psi = -Res$$

$$\underline{\Psi}_{inc}$$

$$\underline{\Psi} = \underline{\Psi_0} \circ \underbrace{\Psi_{inc}}_{incóg}$$

Otras opciones

Conformal Rotation Vector:

$$\underline{c} = 4\,\underline{n}\tan\phi/4 \qquad c_i = \frac{4e_i}{1+e_0}$$

∄ singularidad en $(-\pi, \pi)$

$$\underline{\underline{R}} = \frac{1}{(4-c_0)^2}\left[\left(c_0^2+8c_0-16\right)\underline{\underline{I}} + 2cc^T + 2c_0\tilde{c}\right]$$

Descripciones genéricas

Ang Euler

secuencia de 3 rot elementales:

a) rot (X_3, ψ)

b) rot (X_1, θ)

c) rot (X_3, ϕ)

∃ singularidad!

Ang Bryant

roll, pitch, yaw (navegación)